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SCATTERING BY A SLOT RADIATOR IN A MULTIMODE WAVEGUIDE

by

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INTRODUCTION

The development of the techniques and application of electromagnetic waves in the microwave region is progressing towards the use of higher frequencies. The physical size of the transmission elements employed at present is already of an order of magnitude comparable with a wavelength. For a further increase in operating frequency one could, of course, further reduce the sizes of the transmission elements. However, one would then be faced with difficult problems of production due to tolerance requirements. From a practical standpoint, the sizes of the transmission elements must be in the realm of easy machining and production. These considerations favor the use of transmission elements whose sizes are large in comparison with a wavelength. Because of that the interest in such transmission systems has increased in recent years. Relatively little is known about such transmission elements. The present investigation concerns itself with such transmission elements which, as a results of the large size-to-wavelength ratios, may propagate several modes.

It is well known that in finite regions, or in waveguides, the electromagnetic field can be described in terms of a discrete set of characteristic modes[1], or elementary waves. Each one of these modes for a loss-free guide has an individual propagation constant which is either a real or imaginary function of the geometry and the dimensions of finite region. Up to the present, most of the applications, and therefore the analysis of propagation, have limited themselves to such dimensions that only one of the infinite number of possible characteristic modes has a real propagation constant[2]. All other modes have imaginary propagation constants, and therefore within a certain distance from their source attenuate to a negligible magnitude. Under these conditions the waveguide could propagate only one mode. For this situation it has been shown that there is a complete analogy between the single mode guide and the standard transmission line[3,4]. This analogy has been extremely useful, particularly from the engineering point of view, as it gives an insight into the waveguide based on the wide knowledge of phenomena in the usual transmission lines. The effect of discontinuities in a waveguide can then be considered in light of the known effect of an equivalent localized impedance[4] on the transmission line.

It is worth pointing out that waveguides which propagate several modes, so-called multimode waveguides, have various other applications beside allowing an increase in the operating frequencies. They offer wide band transmission systems[5], possibility of multiplex operation in a single waveguide, mode mixing devices, and associated control of the illumination of horn apertures, etc. In the present work we do not discuss these applications which offer a wide and diverse field for research. We limit ourselves to a particular problem which at the moment is of great practical and theoretical interest.

A basic property of the characteristic modes in a uniform cylindrical waveguide is that of orthogonality; hence there is no energy interchange between the modes[6]. A uniform waveguide allowing the propagation of several modes would then act like a set of independent transmission lines[2], but in the region of a discontinuity in the waveguide, the orthogonality of the modes breaks down. In the region of the obstacle, then there would be an interaction between the modes, which would appear as cross coupling of the otherwise different and independent transmission lines.

The problem of propagation in a waveguide is basically a field problem and, as such, the scattering of waves is the physical aspect which underlines the consideration of an obstacle. Having determined the scattering properties of an obstacle, we can then look for other representations which would have advantages for specific considerations. It is questionable whether the representation of a multimode guide by equivalent transmission lines and obstacles by equivalent localized impedance network has the same merits as in the case of a single mode guide. The theory of multiple transmission lines, coupled by localized multiple networks is far from being highly developed and widely known[7]. Nevertheless, the engineer is used to thinking in terms of circuits and impedances, and such a representation might facilitate the formulation of a physical piture of the phenomena involved. With this in mind, the basis for the equivalent circuit representation will be considered, and application will be made to the particular problem of a slot radiator in a multimode guide.

In considering the problem of a general obstacle we limit ourselves to general considerations of the scattering matrix. Assuming that we know the scattering matrix, we can investigate some of its properties on the basis of general laws that we know about the fields. The determination of the scattering matrix itself involves the solution of the boundary value problem represented by the waveguide with the obstacle in it.

These boundary values problems in most cases are rather tedious. In particular, we are actually faced with two problems:

a) Given a certain source distribution over the obstacle, what would be the fields set up by it, that would satisfy Maxwell's equations and the prescribed boundary conditions; b) Given a certain field propagating in the waveguide, what would be the induced sources on the obstacle. In general, we are given a certain incident field, and to find the effect of a discontinuity we have to determine first the induced sources, and then derive the scattered field produced by these induced sources.

We consider in detail the problem of the slot radiator in a multimode waveguide. The effort involved in solving the above mentioned two boundary value problems is compensated by the wide applications and advantages of slot radiators. In this work an approximate solution to both of these problems is presented. It is shown how to obtain both the amplitude and the distribution of the induced voltage in the slot as a function of the exciting field in the guide and the slot geometry. This step in the theory is extremely important as it facilitates the solution of a great many problems associated with slots. In the single mode guide it has been possible to circumvent solving this aspect of the problem by making judicial assumptions[6] in order to simplify the specific problem. The amplitude of the induced voltage as a function of the exciting field has not been determined explicitly. For the case of a single mode guide one could circumvent it by the use of an energy balance relation[6]. Although in Stevenson's work[9] an expression is given for the amplitude, it involves an infinite series which is difficult to evaluate.

The energy balance relation employed in the single mode guide theory involves a knowledge of the power radiated by the slot[9]. To determine that, we must know, besides the induced voltage which can be eliminated in the single mode case, the radiation resistance. For the radiation resistance the usual procedure has been to employ the external impedance of the slot. This impedance relates the voltage across the slot to the complex power in the exterior region. Under the assumption that the slot is in an infinite perfectly conducting plane this impedance can be evaluated for a given E_{tang} in the slot. There are several ways to do this and they have been discussed in the literature[21]. Often the value employed is the one obtained by the application of the Babinet Principle[10]. For a slot in the wall of a waveguide, this value is then multiplied by a factor of two on the basis of a physical argument. This argument states that, since the waveguide limits the radiation to one direction, the radiated power will be doubled.

Computations on the basis of the above value lead to a rather crude approximation. There is a disagreement of about 35 percent between the theoretical results and experimental data[8]. This is hardly surprising as this procedure neglects the different nature of the fields inside the waveguide. These fields have a specific and definitely different nature than the fields behind a screen in free space. If these assumptions are crude approximations in the case of a single mode guide, they would be worse in the case of a multimode guide with its more complex structure of fields. Further, in such a multimode guide, even if one would be satisfied with this procedure, the circumvention of the required knowledge of the amplitude of the induced source is probably impossible. It is evident therefore that a better approximation is definitely needed. The theory should also provide an answer to the induced voltage problem.

In this work an analytical method developed for wire antennas[11] has been extended and applied to the slot problem. The method is in principal similar to the one outlined in Stevenson's[9] work, but it yields answers in closed form directly applicable. In fact, the results are applied to the multimode guide and give very satisfactory agreement with experimental data. We consider the physical difference between the far zone fields of a slot in an infinite plane, and a slot in the wall of a waveguide. The far zone fields of a slot in an infinite plane satisfy Sommerfield's radiation conditions on both sides of the slot. This tells us that the field amplitudes go to zero as the observation point moves out to infinity. In fact, it prescribes how fast the fields have to go to zero. For the case of a slot in the wall of a waveguide this is true only for the outside region; inside the waveguide the fields do not decrease in amplitude. The radiation condition on the fields is that there will be no reflected waves coming from infinity. The far zone field is just the sum of all the freely propagating modes in the waveguide. We present, therefore, such a description where we can take account of this physical information directly.

It has been customary in the analysis of slots[6,9] to assume that the outside wall of the waveguide forms part of an infinite perfectly conducting plane. This assumption may be one of the reasons for the discrepancy between the theoretical and experimental results. In the present work this same assumption is made, but the theoretical approximation it involves is directly evident. This point is further discussed in Chapter IV of the text.

Outlined below is the approximation method to be followed here. We employ asymptotic approximations for Green's functions in the outside and inside regions. In terms of these approximated Green's functions we express the scattered fields in both regions. To match the fields across the slot we apply the boundary conditions. This leads to an integro-differential equation for the induced sources. From this equation we get a function which is analogous to an admittance function. An approximate evaluation of this rather complicated function is done. It yields as its major part a value that corresponds to external impedance obtained from Babinet's Principle or otherwise. We also get a second term which can be looked upon as a correction term, corresponding to the internal impedance. This correction term, as should be expected, is determined by the freely propagating modes that compose the far-zone field. It is interesting to note that, applying the value of the radiation impedance computed here to the case of a single mode guide gives good agreement with experimental data.

On the basis of the computed induced voltage in the slot, we solve the second part of the boundary value problem. This involves finding the scattering matrix in the multimode guide, which is done by applying the Lorentz theorem in a fashion similar to the one outlined in Silver's book[6]. The extension of this to the multimode guide presents no serious difficulties. Finally the theoretical values are compared with experimental measurements and very good agreement is observed.

Many of the experimental results on the scattering of a slot in a rectangular waveguide propagating TE_{10} and TE_{20} modes, which are used for comparison, were measured in the University of California Antenna Laboratory by W. Kummer before this theoretical work was begun. The writer also took some additional data, using the experimental methods for the excitation and seperation of TE_{10} and TE_{20} modes worked out by Kummer.

CHAPTER I

GENERAL CONSIDERATIONS OF OBSTACLES AND DISCONTINUITIES

We confine the present discussion to waveguides of arbitrary, but uniform cross section. The guide walls are assumed to be perfectly conducting, and the interior of the guide is filled with a lossless medium of dielectric constant and permeability. Under these conditions, one finds that the solution of Maxwell's equations can be presented by a set of transverse modes of two kinds. For one, $E_z = 0$ and for the other, $H_z = 0$ and they are called transverse electric (TE) and transverse magnetic (TM) respectively[6]. For TE modes

$$\underline{H} = e^{-\gamma z}\left(-\frac{\gamma}{K^2_{m,n}} \nabla u + u \cdot \underline{i}_z\right) \tag{1.1}$$

and

$$\underline{E} = \frac{\omega\mu}{j\gamma}\left(H \times \underline{i}_z\right) \tag{1.2}$$

where $U(xy)$ is a solution of the differential equation

$$\nabla_t^2 u + K^2_{m,n} u = 0 \tag{1.3}$$

and

$$\gamma_{mn} = \left(K^2_{mn} - k^2\right)^{\frac{1}{2}} \quad ; \quad k^2 = \omega^2\mu\varepsilon$$

K_{mn} are the characteristic values associated with the set of orthonormal eigenfunctions $U_{mn}(xy)$ corresponding to the cross section of the guide. Their determination allows for a multiplicative constant, which we shall choose so as to obtain a convenient normalization. We can treat in a similar manner the TM modes[6].

If we normalize all the modes so that the power flow P across a cross section of the guide is unity we have

$$P = \frac{1}{2}\int_{\sigma}\int Re\,(\underline{E} \times \underline{H}) \cdot \underline{n}\, d\sigma = 1 \tag{1.4}$$

Substituting from (1.1) and (1.2) for the propagating modes $\gamma = j\beta$ we find

$$P = \frac{1}{2} Re \int_{\sigma}\int \left[\frac{\omega\mu}{-\beta}(\underline{H} \times i_z) \times \underline{H}^*\right]_z d\sigma \tag{1.5}$$

we find

$$P = \frac{\omega\mu\beta_{mn}\alpha^2}{2K_{mn}^2 K_{mn}^{*2}} \int_{\sigma}\int (\nabla u)^2 d\sigma = 1$$

hence

$$\alpha = \sqrt{\frac{2}{\omega\mu\beta_{mn}}}\ \left|K_{mn}^2\right| \tag{1.6}$$

as the eigenfunctions are normalized so that

$$\int_{\sigma}\int (\nabla u)^2 d\sigma = 1$$

Let us denote now waves propagating to the right and left with a superscript of (-) and (+) respectively

to right | to left

$$\underline{E}_i^- = \underline{\psi}_i^-(xy)e^{-\gamma z} = \underline{e}_i^- \qquad \underline{E}_i^+ = \underline{\psi}_i^+(xy)e^{+\gamma z} = \underline{e}_i^+$$

$$\underline{H}_i^- = \underline{\varphi}_i^-(xy)e^{-\gamma z} = \underline{h}_i^- \qquad \underline{H}_i^+ = \underline{\varphi}_i^+(xy)e^{+\gamma z} = \underline{h}_i^+ \tag{1.7}$$

Every field in the waveguide can be represented as a linear combination of these orthonormal base vectors e_i and h_i.

For the TE modes the vector functions ψ (xy) and φ (xy) are given below explicitly.

$$\psi_{ix}^{\pm} = -j\sqrt{\frac{2\mu}{\beta_i\omega}}\ \frac{\partial u_i}{\partial y} \qquad \varphi_{ix}^{\pm} = \pm j\sqrt{\frac{2\beta_i}{\omega\mu}}\ \frac{\partial u_i}{\partial x}$$

$$\psi_{iy}^{\pm} = j\sqrt{\frac{2\mu}{\beta_i\omega}}\ \frac{\partial u_i}{\partial x} \qquad \varphi_{iy}^{\pm} = \pm j\sqrt{\frac{2\beta_i}{\omega\mu}}\ \frac{\partial u_i}{\partial y}$$

$$\psi_{iz}^{\pm} = 0 \qquad \varphi_{iz}^{\pm} = \sqrt{\frac{2}{\omega\mu\beta_i}}\ K_i^2\, u_i \tag{1.8}$$

Suppose we have an incident wave coming from infinity in region 1 (Fig. 1, page 69) and an obstacle at $z = 0$ of known scattering properties. We denote by $S_{ki}^{12}\, a_i^1$ the amplitude of the k^{th} mode in region 2 due to the i^{th} mode of the amplitude a_i^1 incident on the obstacle in region 1. This means that S_{ki}^{12} for example is the scattering coeficient giving the transfer of energy from left to right of obstacle. If a wave of mode i will be incident on the obstacle in both regions 1 and 2 with amplitude a_i^1 and a_i^2 respectively, the fields due to it will be for $z < 0$

$$\underline{E}^i(xyz) = a_i^1 e_i^- + \sum_{K=1}^{\infty} S_{Ki}^{11} a_i^1 e_{-K}^+ + \sum_{K=1}^{\infty} S_{Ki}^{21} a_i^2 e_{-K}^+$$

$$\underline{H}^i(xyz) = a_i^1 \cdot \underline{h}_i^- + \sum_{K=1}^{\infty} S_{Ki} a_i^1 e_{-K}^- + \sum_{K=1}^{\infty} S_{Ki}^{22} a_i^2 e_{-K}^- \qquad (1.9)$$

and for $z > 0$

$$\underline{E}^i(xyz) = a_i^2 e_i^+ + \sum_{K=1}^{\infty} S_{Ki}^{12} a e_{-K}^- + \sum S_{Ki}^{22} a_i^2 e_{-K}^-$$

$$\underline{H}^i(xyz) = a_i^2 h_i^+ + \dots$$

(1.10)

We obtain a similar expression for H^i except and h_i^- replaces e_i^-.

In the case of an arbitrary incident field the total field will be the sum of the contributions of all the modes in the incident field. We can write th for $z < 0$

$$\underline{E}_1(xyz) = \sum_{i=1}^{\infty} \sum_{K=1}^{\infty} \left[\delta_{iK} a_i^1 \underline{e}_K^- + \left(S_{Ki}^{11} a_i^1 + S_{Ki}^{21} a_i^2 \right) \underline{e}_K^+ \right] \qquad (1.11)$$

and for $z > 0$

$$\underline{E}_2(xyz)=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\left[\delta_{ik}a_i^2\underline{e}_k^+ + (S_{ki}^{12}a_i^1 + S_{ki}^{22}a_i^2)\underline{e}_k^-\right] \qquad (1.12)$$

and similar expressions for H.

Since we can also describe the total fields in terms of the orthonormal base vectors e_i; h_i we can write

$$\underline{E}_1(xyz)=\sum_i a_i^1\underline{e}_i^- + b_i^1\underline{e}_i^+$$

$$\underline{H}_1(xyz)=\sum_i a_i^1\underline{h}_i^- + b_i^1\underline{h}_i^+ \qquad (1.13)$$

and

$$\underline{E}_2(xyz) = \sum_i a_i^2 \underline{e}_i^+ + b_i^2\underline{e}_i^-$$

$$\underline{H}_2(xyz) = \sum_i a_i^2 \underline{h}_i^+ + b_i^2\underline{h}_i^- \qquad (1.14)$$

comparing (1.13) and (1.14) to (1.11) and (1.12) we get

$$b_K^1 = \sum_i S_{Ki}^{11} a_i^1 + S_{Ki}^{21} a_i^2$$

$$b_K^2 = \sum_i S_{Ki}^{12} a_i^1 + S_{Ki}^{22} a_i^2 \qquad (1.15)$$

Upon changing the order of numbering so that index 2 on top starts after index N on bottom, where N is the number of propagating modes we can write in matrix form

$$(B) = (S) \quad (A) \qquad (1.16)$$

where (B) and (A) are the matrices of the reflected and incident amplitudes respectively. Therefore by definition[2] (S) is the scattering matrix. We investigate now the properties of the element S_{ik} of this matrix.

To investigate the properties of S_{ik} we consider a region that includes the obstacle and is bounded by the inner walls, and two cross section at $z = z_1 < 0$ and $z = z_2 > 0$. Since we consider a source-free region, the divergence of the field vectors is zero. Consider two fields $E^{(m)}$, $H^{(m)}$ and $E^{(n)}$ $H^{(n)}$ of the same frequency, that satisfy Maxwell's equations. Then we can apply the Lorentz theorem which states

$$\int_\sigma \int (\underline{E}^{(m)} \times \underline{H}^{(n)} - (E^{(n)} \times H^{(m)}) \cdot \underline{n} \, d\sigma = 0 \qquad (1.17)$$

In this region we apply the theorem to two fields due to particular modes set up in particular ways to give the desired relationships between the elements of the scattering matrix (S).

Consider first the field due to mode 'm' of unit amplitude incident from left and mode 'n' of unit amplitude incident from right, and express these fields by expressions similar to (1.11) and (.12). Then substitute the above in (1.17) and make use of the normalization and orthogonality properties of the eigenvectors e_v and h_v to obtain

$$2\left(S_{mn}^{21} - S_{nm}^{12}\right) + \iint_{walls} \left(\underline{E}^{(m)} \times \underline{H}^{(n)} - \underline{E}^{(n)} \times \underline{H}^{(m)}\right) \cdot \underline{n}\, d\sigma = 0 \qquad (1.18)$$

We consider now three general cases.

<u>Case a</u> - Lossless obstacles (perfectly conducting) in a closed waveguide.

In this case the boundary conditions on $E^{(m)}H^{(m)}$ and $E^{(n)}H^{(n)}$

$$\begin{aligned} \underline{n} \times \underline{E} &= 0 \\ \underline{n} \cdot \underline{H} &= 0 \end{aligned} \qquad (1.19)$$

tell us immediately that the integral in (1.18) vanishes. Hence $S_{mn}^{21} = S_{nm}^{12}$.

<u>Case b</u> - Lossy obstacles (dielectrics etc.) in a closed waveguide.

In this case we change our volume and surface integration. The integrals over the perfectly conducting part of the wall vanish as in case a, and we are left with an integral over outer surface of obstacle. We apply the same theorem to the volume of the obstacle. The surface of integration will now be the inner surface of the obstacle, see Fig. 2, page . We have then

$$\int_{\sigma_{\text{inner surface}}} \left(\underline{E}^{(m)} \times \underline{H}^{(n)} - \underline{E}^{(n)} \times \underline{H}^{m}\right) \cdot \underline{n}\, d\sigma = \int_{\sigma} \underline{F}_{in} \cdot \underline{n}\, d\sigma = 0 \qquad (1.20)$$

and the integral over inner surface vanishes. Now, since in expression (1.20) only the tangential component of E and H contribute, and by the boundary conditions we know that the tangential components go over continuously across the boundary, we have

$$F_{inside} = F_{outside} \qquad (1.21)$$

and therefore we are left with

$$S^{21}_{mn} - S^{12}_{nm} = 0 \tag{1.22}$$

in this case also.

Case c - Radiating Obstacles - Slots in Waveguide Wall

In this case, applying the theorem, we will be left with and integral over inner boundary of the slot.

$$\int_{\sigma_{SLOT}} \underline{F}_{in} \cdot \underline{n}\, d\sigma \tag{1.23}$$

We use now a similar technique to the one used in case b. We apply the Lorentz theorem to the outside region, see Fig. 3 page . We have now

$$\int_{SPHERE\ R} \underline{F} \cdot \underline{n}\, d\sigma + \int_{SLOT} \underline{F}_{OUT} \cdot \underline{n}\, d\sigma + \int_{\sigma\ of\ G} \underline{F} \cdot \underline{n}\, d\sigma = 0 \tag{1.24}$$

As in case a, over G the integral goes to zero due to conditions (1.19). We consider now the case where the Radius of the sphere R goes to infinity.

$$\int_{\infty} \underline{F} \cdot \underline{n}\, d\sigma = \lim_{R \to \infty} \int \underline{F} \cdot \underline{R}\, dR \tag{1.25}$$

and as F is a combination of the radiation fields due to the respective modes, it will satisfy the Sommerfeld radiation conditions. We obtain then

$$\int_{\sigma_{SPHERE}} \underline{F}\, \underline{n}\, d\sigma = 0$$

and consequently

$$\int_{\substack{\sigma \\ SLOT}} \underline{F}_{OUT} \cdot \underline{n} \, d\sigma = 0 \tag{1.26}$$

By the same argument about the continuity of the tangential component as in case b, we see that

$$\int_{\sigma} \underline{F}_{in} \cdot \underline{n} \, d\sigma = \int_{\sigma} \underline{F}_{OUT} \cdot \underline{n} \, d\sigma = 0 \tag{1.27}$$

hence we conclude the symmetry of the respective elements of (S) for any general obstacle.

Applying the same argument as outlined in deriving (1.18), to the field due to mode m and n both incident from either left or right, we get the required relation for all the necessary elements. This establishes the symmetry properties of the scattering matrix S.

For lossless obstacles we can also show that the matrix (S) will be unitary(2). That is

$$\sum_{K=1}^{n} S_{KV} S_{Ki} = \delta_{Vi} \tag{1.28}$$

where

$$\delta_{Vi} = \begin{cases} 0 & \text{FOR } V \neq i \\ 1 & \text{FOR } V = i \end{cases}$$

This can be done by applying the theorem of conservation of energy. Expressing the fields due to any particular mode by expressions similar to (1.11) and (1.12) and substituting in the expressions for Poynting's vector, we obtain the desired result.

EQUIVALENT REPRESENTATION:

If we are to look upon the waveguide as a set of transmission lines, we must be able to represent the discontinuity by the impedance matrix (Z), where (Z) relates the equivalent voltages and currents as described in the literature[2] through Ohm's law, (V) = (Z) (I). Since the system is bilateral, linear and isotropic, the matrix Z will have to be symmetrical, i.e., $Z_{ik} = Z_{ki}$. If (S) is the scattering matrix then the impedance matrix can be given by[2]

$$Z = \left((1) - (S)\right)^{-1} \left((1) + (S)\right) \tag{1.29}$$

It can be readily shown* that if $S_{ik} = S_{ki}$ then $Z_{ik} = Z_{ki}$ and conversely.

The symmetry of (S) has been proven above for any arbitrary kind of obstacle. This establishes, then, the validity of representing a multimode waveguide with discontinuities by a system of transmission lines with certain coupling networks between the lines.

To find the equivalent representation we have to find the matrix (S). This involves solving the boundary value problem. Then from (S) and (1.29) we obtain the matrix (Z). We proceed now to the solution of the problem of an obstacle in the form of a narrow slot radiator in a cylindrical waveguide.

*Write $Z = L^{-1}M$, post multiply by M, use symmetry of L and M to obtain symmetry of Z.

CHAPTER II

THE INDUCED SOURCES

To compute the voltage induced in a slot cut in the wall of a waveguide we have to consider two different regions. One is the inside of the waveguide and the other is the external space, bounded by the outer walls of the waveguide and a sphere at infinity. The slot is common to both regions, and the solutions for both regions have to be matched in the slot.

Starting out with an arbitrary distribution of electric field in the slot, we determine the vector potential of this source in both the outside and inside regions. The vector potential is expressed in terms of Green's functions for the two regions. For these functions we employ asymptotic approximations discussed in Chapter IV. Having determined the vector potential we can find the associated induced magnetic fields in the two regions. The expression for the fields will depend, of course, on the electric field in the slot. Applying the boundary condition on the continuity of the tangential component of the magnetic field across the slot, we get a condition on the arbitrary " sources" in the slot. This condition determines the proper induced sources as a function of the geometry of the slot, waveguide, and the incident field. The incident field comes in, as the tangential component of the magnetic field inside is the sum of the induced (scattered) and incident fields.

FORMULATION OF PROBLEM

Consider a cylindrical waveguide and a slot cut in the wall. If the slot perturbs the current distribution that would exist on the wall of the guide for a certain given electromagnetic field inside the waveguide there will be a leakage of the inside field to the outside. We then say that there is radiation through the slot in the wall. To determine the radiation of the slot a basic problem is that of finding the field excited in the slot, given the fields in the enclosed region before the slot was cut out. Let us consider a slot in the top wall of rectangular guide that is cut parallel to the z direction. The slot and the coordinates are shown in Fig. 4. In formulating the problem, we follow the procedure of Stevenson[9].

We have to consider two regions, one inside the perfect conductor, and one outside. Assuming a time dependance of the form $e^{j\omega t}$, and that the two region are filled with the same dielectric material, we will have for both regions

$$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$
$$\nabla \times \underline{H} = j\omega\epsilon \underline{E} \tag{2.1}$$

where $\underline{E}$ and $\underline{H}$ are the electric and magnetic field vectors respectively. As there are no sources in the region we also have

$$\nabla \cdot \underline{E} = 0$$
$$\nabla \cdot \underline{H} = 0 \tag{2.1a}$$

If we take the curl of the second equation in (2.1) we have

$$\nabla \times \nabla \times \underline{H} = -\nabla^2 \underline{H} + \nabla \nabla \cdot \underline{H} = j\omega\epsilon \nabla \times \underline{E} \tag{2.2}$$

By (2.1a) and (2.1) this reduces to

$$\nabla^2 \underline{H} + k^2 \underline{H} = 0 \tag{2.3}$$

where $\quad k^2 = \omega^2 \mu\epsilon = \dfrac{\omega^2}{c^2}$

Let us now consider a rectangular waveguide, in which we have a system of rectangular coordinates (x,y,z), where the z coordinate is parallel to the generating line of the cylinder. For the z component of the magnetic field

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} + k^2 H = 0 \tag{2.4}$$

In order to be able to solve this differential equation we have to find the boundary conditions for H_z. From Maxwell's equation (2.1) we have

$$H_z = -\frac{1}{j\omega\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \tag{2.5}$$

and

$$\frac{\partial H_z}{\partial y} = -\frac{1}{j\omega\mu_0}\left(\frac{\partial E_y}{\partial x \partial y} - \frac{\partial^2 E_x}{\partial y^2}\right) \tag{2.5a}$$

Since in a similar manner to that used in deriving (2.3) we can show that

$$\nabla^2 \underline{E} + k^2 \underline{E} = 0 \tag{2.3a}$$

we have

$$-\frac{\partial^2 E_x}{\partial y^2} = \left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x + \frac{\partial^2 E_x}{\partial x^2} \tag{2.6}$$

and substitution in (2.5a) gives us

$$\frac{\partial H_z}{\partial y} = -\frac{1}{j\omega\mu_0}\left[\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x + \frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right)\right] \tag{2.6a}$$

which can be written as

$$\frac{\partial H_z}{\partial y} = -\frac{1}{j\omega\mu_0}\left[\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_x + \frac{\partial}{\partial x}\left(\nabla \cdot E - \frac{\partial E_z}{\partial z}\right)\right]$$

Thus, on the boundaries of the region the z component of the magnetic field will satisfy the condition

$$\frac{\partial H_z}{\partial Y} = -\frac{1}{j\omega\mu_0}\left[\left(\frac{\partial^2}{\partial Z^2} + k^2\right)E_x - \frac{\partial^2 E_z}{\partial X\,\partial Z}\right] \tag{2.7}$$

It is evident that in our coordinates $\frac{\partial H_z}{\partial Y}$ is the normal derivative of this component of the magnetic field. On a perfect conductor, the tangential components of E are zero. Therefore E_x and E_z are zero on the walls and $\frac{\partial H_z}{\partial Y} = 0$ everywhere on the walls except in the slot itself.

By a well known relation in the theory of Green's functions(13)*, the differential equation for H_z(2.4), with the boundary conditions (2.7), can be solved in the following form

$$H_z = \frac{1}{j\omega\mu_0}\iint_\sigma G(PP')\left[\left(\frac{\partial^2}{\partial Z^2} + k^2\right)E_x(P') - \frac{\partial^2 E_z}{\partial X\,\partial Z}\right]dx'dz' \tag{2.8}$$

By repeated integration by parts, this can be transformed to give

$$H_z(P) = \frac{1}{j\omega\mu_0}\iint_\sigma \left(\frac{\partial^2}{\partial z^2} + k^2\right) G(PP')\,E_x(P') - \frac{\partial^2 G(P,P')}{\partial x'\,\partial z'}E_z(P')\,dx'dz' \tag{2.9}$$

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*In connection with this point it may be worth pointing out that in the case of waveguide of arbitrary shapes (not cylindrical) we would have to use dyadic Green functions to describe the fields due to sources. This is described in a paper by H. Lavine and J. Schwinger published by the symposium on electromagnetic theory (1950). As was pointed out to the author by Prof. S. Silver, this is due to the fact that the boundary conditions are then neither of the Dirichlet type nor of the Neuaman type, but a combination of the two. However, in a cylindrical waveguide we can describe the vector fields in terms of two Green's functions, as was done by Stevenson(9). In this particular case one obtains boundary conditions for E_z and H_z as given in Stevenson's paper, formulae (3), (4) and (5). When we limit ourselves to slots that are parallel to the generating line of the cylinder, as we have done here, only a Neuaman kind of Green function is necessary, as we have currents in the z direction only. The author wishes to thank Prof. M. Schiffer for his help in clarifying these points.

The Green's function $G(P,P')$ should satisfy the following:

a) The homogeneous wave equation except at a single point, $P = P'$.
b) radiation conditions at large distances,
c) boundary conditions on walls of conductors in the relevant geometry, i.e. $\frac{\partial G}{\partial n} = 0$ (2.10)
d) have a singularity of the type $\frac{e^{-jkR}}{R}$ when the point of observation converges on the source point ($\lim. R \to 0$).

Any function $G(P,P')$ that will satisfy all the above mentioned conditions will be a perfectly good Green's function when multiplied by the proper normalization factor. Further, all the field expressions are linear; therefore, we can break up the function G into linear combinations. In particular we can put

$$G(P,P') = G_1(P,P') + g(P,P') \tag{2.11}$$

If $G_1(P,P')$ is known, being the Green's function for some known simple geometry, then the conditions for the determination of $g(P,P')$ are such as to make $G(P,P')$ satisfy all conditions (2.10) in the region of interest. For free space divided into two half spaces by a perfectly conducting screen we know the Green's function rigorously. We shall use this function, which is given by

$$G_1(P,P') = \frac{1}{2\pi} \frac{e^{-jkR}}{R} \tag{2.12}$$

as the function $G_1(P,P')$ in (2.11).

This function satisfies condition (2.10) 'a' and 'd'. We are interested in finding the Green's function in two different regions; the region of space constituting the outside of the waveguide and the region of space inside it. For both regions we should determine the functions $g(P,P')$, but this in itself is a considerable task. In Chapter IV we shall discuss some approximations for these functions.

THE INSIDE REGION

For the inside region we write

$$G(P, P') = \frac{1}{2\pi} \frac{e^{-jkR}}{R} + g^{i}(P, P') \tag{2.13}$$

where $g^{i}(P,P')$ is to be a function that satisfies condition (2.10a), is regular everywhere, and is selected so as to have $G(P,P')$ satisfy the radiation conditions inside, (see introduction), and $\frac{\partial G(P,P')}{\partial m} = 0$ on the perfectly conducting walls. In terms of the Green's function we can, by (2.9) write

$$H_z(P) = \frac{1}{j\omega\mu_0} \Big\{ \iint \Big[\Big(\frac{\partial^2}{\partial z^2} + k^2\Big) \frac{e^{-jkR}}{2\pi R} - \frac{\partial^2}{\partial z \partial x} \frac{e^{-jkR}}{2\pi R} \cdot E_z \Big] dx'dz'$$

$$+ \iint \Big[\Big(\frac{\partial^2}{\partial z^2} + k^2\Big) g^{i} E_x - \frac{\partial^2 g^{i}}{\partial z' \partial x'} E_z \Big] dx'dz' \Big\} \tag{2.14}$$

where the surface of integration in (2.14) is only over the surface of the slot. Let us now introduce a local system of coordinates in the slot, (see Fig. 5) and consider the electric field components in the slot. In (2.14) $E_x(x,z)$ will correspond to $E_{\eta}(\eta, \zeta)$ and $E_z(x,z)$ will correspond to $E_{\zeta}(\eta, \zeta)$. We introduce now the following assumption about the slot:

a) the wavelength is large in comparison with the sidth, $\frac{2d}{\lambda} << 1$.

b) The slot is narrow in comparison with its length, $\frac{d}{l} << 1$.

With these assumptions we can safely assume that the variation of E_{ζ} across the slot will not be appreciable. Becauseof continuity of E_{ζ} and the fact that at the edges this component disappears, $E_{\zeta}(\pm d, \zeta) = 0$ and $2d << \lambda$, we neglect E_{ζ} in comparison with $E_{\eta}(\eta, \zeta)$. It is to be noted that these arguments are on the basis of physical plausibility. Theoretically this is an assumption. It should be noted that it is analogous to the approximations made in the thin wire antenna theory[11] concerning the current over the cross section of the wire.

On the basis of this assumption and the local coordinates we can rewrite (2.14) and get

$$H_z(P)=\frac{1}{j\omega\mu_0}\Big[\iint\Big(\frac{\partial^2}{\partial z^2}+k^2\Big)\frac{e^{-jkR}}{2\pi R}E_\eta(\eta,\xi)\,d\eta\,d\xi+ \iint\Big(\frac{\partial^2}{\partial z^2}+k^2\Big)g^i(P,P')E_\eta\,d\xi\,d\eta\Big] \tag{2.14a}$$

In the two integrals in (2.14a), the field component $E_\eta(\eta,\xi)$ is assumed to be continuous and to possess continuous derivative, so we can change the order of integration and differentiation, and write

$$H_z(P)=\frac{1}{j\omega\mu_0}\Big(\frac{\partial^2}{\partial z^2}+k^2\Big)A_\xi \tag{2.15}$$

where

$$A_\xi=\frac{1}{2\pi}\int_{-l}^{l}\int_{-d}^{d}E_\eta(\eta,\xi)\,\frac{e^{-jkR}}{R}\,d\eta\,d\xi+ \int_{-l}^{l}\int_{-d}^{d}E_\eta(\eta,\xi)\,g_i(P,P')\,d\eta\,d\xi \tag{2.16}$$

We proceed now to transform A_ξ into a form that will lead to a formulation that can be solved. This step involves making some approximations, and it is indeed dangerous to approximate before a differentiation. However, the approximations that we have to make are with respect to the dependance on the x coordinate, whereas the differentiation is with respect to the z coordinate. Nevertheless, the validity of the approximations should be checked. One way, of course, is to see that the results are consistent, and how well they describe the known physical phenomena.

It may be worth pointing out that by considering an equivalent magnetic current on the slot(12), and introducing a magnetic vector potential, we are led to a result similar to (2.15). In doing that we have to make similar approximations to those described above, and we find then that A_ζ of (2.16) is the ζ component of the vector potential.

Having assumed a norrow slot so that $2d \ll \lambda$ and $2d \ll 2\ell$, assume that the variation of the Green's functions with η is negligible in comparison to the variation with ζ. We can separate the integration with respect to η and ζ, and write

$$A_\zeta = \frac{1}{2\pi}\int_{-\ell}^{+\ell} V(\zeta')\,\frac{e^{-jkR}}{R}\,d\zeta' + \int_{-\ell}^{+\ell} V(\zeta')\,g^i(\eta', b\zeta';\ xyz)\,d\zeta' \tag{2.17}$$

where $V(\zeta)$ is by definition the voltage across the slot

$$V(\zeta') = \int_{-d}^{d} E_\eta(\eta', \zeta')\,d\eta' \tag{2.18}$$

Let us now separate out the principal part of the first integral in expression (2.16). As we are interested in the fields in the slot itself, we consider observation points in the slot. For these points we can write

$$R^2 = (\zeta - \zeta')^2 + \rho^2 \tag{2.19}$$

where

$$\rho^2 = (\eta - \eta')^2 + (y - b)^2$$

Since we are interested in the fields in the slot itself we have to consider points P very close to P'. At P = P' we have a point of singularity. Separating to exclude the singularity we write

$$\int_{-l}^{l} = \lim_{\varepsilon \to 0} \left(\int_{-l}^{\zeta-\varepsilon} + \int_{\zeta+\varepsilon}^{l} \right)$$

where $\varepsilon > 0$ and arbitrarily small.
Now for $\zeta' < \zeta$ we have from (2.19)

$$\zeta - \zeta' = +\sqrt{R^2 - \rho^2}$$

$$\zeta' = \zeta - \sqrt{R^2 - \rho^2}$$

$$d\zeta' = -\frac{R\,dR}{\sqrt{R^2 - \rho^2}} \qquad (2.20)$$

and for $\zeta' > \zeta$

$$\zeta - \zeta' = -\sqrt{R^2 - \rho^2}$$

$$\zeta' = \zeta + \sqrt{R^2 - \rho^2}$$

$$d\zeta' = \frac{R\,dR}{\sqrt{R^2 - \rho^2}} \qquad (2.20')$$

We get therefore the principal part of A

$$A_\varsigma = \frac{1}{2\pi}\int_{-\ell}^{\varsigma=\varsigma'} v(\varsigma')\,\frac{e^{-jkR}}{R}\cdot\frac{-R}{\sqrt{R^2-\rho^2}}\,dR + \int_{\varsigma=\varsigma'}^{\ell} v(\varsigma)\,\frac{e^{-jkR}}{R}\,\frac{R}{\sqrt{R^2-\rho^2}}\,dR + \int_{-\ell}^{\ell} v(\varsigma)\,g'\,d\varsigma'$$

(2.21)

$$A_\varsigma = \frac{1}{2\pi}A' + \int_{-\ell}^{\ell} v(\varsigma')\,g(P,P')\,d\varsigma'$$

and integrating A' by parts we find

$$A' = -ve^{-jkR}\ln k\left(R+\sqrt{R^2-\varsigma^2}\right)\Big|_{-\ell}^{\varsigma=\varsigma'} + ve^{-jkR}\ln k\left(R+\sqrt{R^2-\varsigma^2}\right)\Big|_{\rho=\rho'}^{\ell}$$
$$+\int_{-\ell}^{\varsigma=\varsigma'}\ln k\left(R+\sqrt{R^2-\varsigma^2}\right)\left[\frac{\partial v}{\partial\rho'}\frac{\partial\rho'}{\partial R}-jkv\right]e^{-jkR}\,dR -$$
$$\int_{\varsigma=\varsigma'}^{\ell}\ln k\left(R+\sqrt{R^2-\rho^2}\right)\left[\frac{\partial v}{\partial\rho'}\frac{\partial\varsigma'}{\partial R}-jkv\right]e^{-jkR}\,dR$$

(2.22)

Introducing the boundary conditions at the edges $V(\pm\ell) = 0$, and $R = \varsigma$ at $\varsigma = \varsigma'$ by (2.19), we can rewrite equation (2.22), and get, after changing back the variable of integration from R to ; the following

$$A' = -2V(\zeta)\ln k\rho e^{-jk\rho} +$$
$$+\left(\int_{-\ell}^{\zeta} - \int_{\zeta}^{\ell}\right)\ln k\left(k+\sqrt{R^2-\rho^2}\right)\left[V' - jkV\frac{\partial R}{\partial \zeta'}\right]e^{-jkR}\,d\zeta' \qquad (2.22')$$

From (2.19) $\frac{\partial R}{\partial \zeta} = \frac{\zeta' - \zeta}{R}$ so we get

$$A' = -2V(\zeta)\ln k\rho e^{-jk\rho} + \left(\int_{-\ell}^{\zeta} - \int_{\zeta}^{\ell}\right)\ln k\left(k+\sqrt{R^2-\rho^2}\right)\left(V' - jkV\frac{\zeta'-\zeta}{R}\right)e^{-jkR}\,d\zeta' \qquad (2.23)$$

Since the first term on the right side of (2.23) is for values in the plane $\zeta' = \zeta$, and of that in the slot itself, (i.e. y=b) the values of ρ are very nearly equal to η' . Further, we have agreed to neglet terms of order $\frac{2d}{\lambda}$, and as the maximum value of ρ is ρ = 2d we can equate $e^{-jk\rho}$ to unity with an error at the most of $\frac{2d}{\lambda} \ll$ 1. We find therefore

$$A' = -2V(\zeta)\ln k\eta + \left(\int_{-\ell}^{\zeta} - \int_{\zeta}^{\ell}\right)\ln k\left(R+\sqrt{R^2-\rho^2}\right)\left(V'(\zeta') - jkV(\zeta')\frac{\zeta'-\zeta}{R}\right)e^{-jkR}\,d\zeta' \qquad (2.23')$$

and

$$A_\zeta(\zeta) = \frac{1}{2\pi}A' + \int_{-\ell}^{\ell} V(\zeta')\, g'(\eta'\zeta', \eta\zeta)\, d\zeta' \qquad (2.24)$$

Let us now multiply this expression by a function $f(\eta)$ and integrate again from $-d$ to $+d$ with respect to η. Also impose the following condition on $f(\eta)$.

$$\int_{-d}^{d} f(\eta)\,d\eta = 1$$

$$\int_{-d}^{d} f(\eta) \ln k|\eta|\,d\eta = \Omega = \ln kd^{(9)} \qquad (2.25)$$

The function $f(\eta)$ governs the distribution across the slot. Such an exists, and one case is

$$f(\eta) = \frac{1}{\pi\sqrt{d^2-\eta^2}}$$

It can be shown easily that it satisfies the conditions (2.25). This function is directly related to the electrostatic potential over a slot in an infinitely thin screen[14]. Hence it relates to the case of infinitely thin walls of the guide. Other functions $f(\eta)$ are possible, such as for the case of finite thickness, one which relates to the corresponding electrostatic problem. We get then

$$A_\zeta(\zeta) = -\frac{2}{2\pi} V(\zeta) \ln kd + \frac{1}{2\pi} \mathcal{U}[V(\zeta), \zeta] + P_i[V(\zeta), \zeta] \qquad (2.26)$$

where

$$\mathcal{U}[V(\zeta), \zeta] = \left(\int_{-l}^{\zeta} - \int_{\zeta}^{l}\right) \ln k\left(R + \sqrt{R^2-\eta^2}\right)\left(V' - jkV\,\frac{\zeta'-\zeta}{R}\right) e^{-jkR}\,d\zeta'$$

and

$$P_i\,[V(\varsigma),\varsigma] = \int_{-\ell}^{\ell} V(\varsigma')\, g_i\,(\eta\varsigma', \eta\varsigma)\, d\varsigma'$$

We find therefore, by substituting (2.26) in (2.15)

$$H_{\mathfrak{z}_i}^{(s)} = \frac{1}{j\omega\mu\, 2\pi}\left\{-2\pi\left(\frac{\partial^2}{\partial\varsigma^2} + k^2\right)V(\varsigma) + \left(\frac{\partial^2}{\partial\varsigma^2} + k^2\right)\left(U\,[V(\varsigma),\varsigma] + 2\pi P_i\,[V(\varsigma),\varsigma]\right)\right\} \tag{2.27}$$

in terms of the local coordinates. (See Fig. 5).
This is the field set up inside the waveguide over the slot by an arbitrary voltage distribution.

THE OUTSIDE REGION

Let us consider now the region outside. The Green's function will be

$$G(P,P') = \frac{1}{2\pi}\,\frac{e^{-jkR}}{R} + g_e(P,P') \tag{2.28}$$

The function $\frac{1}{2\pi}\,\frac{e^{-jkR}}{R}$ satisfies conditions (2.10) 'a' 'b' and 'd'. The function $g_e(P,P')$ has to be determined so that it satisfies condition (2.10) 'a' and makes $G(P,P')$ satisfy condition (2.10) 'c' on outer walls of waveguide.

For the outside region we find then that

$$A = \frac{1}{2\pi}\int_\sigma E_\eta(\eta'\varsigma')\,\frac{e^{-jkR}}{R}\,d\sigma + \int_\sigma E_\eta(\eta_1'\varsigma_1')\,g_e(\eta'\,\&\,\varsigma';\,xyz)\,d\sigma \tag{2.29}$$

A change in sign comes in as we keep the same direction for the normal. Carrying through for A the same transformations as for the corresponding A in the inside region we finally get

$$A_\varsigma = \frac{2}{2\pi} V(\varsigma)\Omega - \frac{1}{2\pi} \mathcal{U}[V(\varsigma),\varsigma] - P_e[V(\varsigma)\varsigma] \tag{2.30}$$

and for the scattered magnetic field over the slot outside we find

$$H_{z_e}^{(s)} = \frac{1}{j\omega\mu_0 2\pi}\left\{2\Omega\left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)V(\varsigma) - \left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)\left(\mathcal{U}[V(\varsigma),\varsigma] + 2\pi P_e[V(\varsigma),\varsigma]\right)\right\} \tag{2.31}$$

DETERMINATION OF PROPER VOLTAGE

We impose now the boundary conditions on tangential H in order to get a condition on the arbitrary voltage distribution we assumed. This will determine the voltage distribution we are seeking. Tangential H is continuous through the opening. Hence

$$H_{z_i}^{o} + H_{z_i}^{(s)} = H_{z_e}^{(s)} \tag{2.32}$$

where H_{zi}^{o} is the magnetic component of the incident unperturbed field.

Substituting for $H^{(s)}$ from (2.27) and (2.31) we get

$$H_z^{o} + \frac{2\Omega d}{2\pi\omega\mu_0}\left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)V(\varsigma) + \frac{1}{j2\pi\omega\mu_0}\left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)(\mathcal{U}+2\pi P_i)$$

$$= -\frac{2\Omega j}{2\pi\omega\mu_0}\left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)V(\varsigma) - \frac{1}{j2\pi\omega\mu_0}\left(\frac{\partial^2}{\partial\varsigma^2}+k^2\right)(\mathcal{U}+2\pi P_e) \tag{2.33}$$

Rearranging we get

$$\left(\frac{\partial^2}{\partial \zeta^2}+k^2\right)V(\zeta)=\frac{\pi\omega\mu}{j2\Omega}\Big[-H_z^\circ-\frac{1}{j\pi\omega\mu}\left(\mathcal{U}''+k^2\mathcal{U}\right)-$$
$$\frac{1}{j\omega\mu}\left(P_i''+k^2P_i+P_e''+k^2P_e\right)\Big] \tag{2.34}$$

$$\frac{\partial^2 V}{\partial \zeta^2}+k^2V=\frac{j\pi\omega\mu}{2\Omega}\left\{H_z^\circ+K[V(\zeta),\zeta]\right\} \tag{2.34'}$$

where

$$K[V(\zeta),\zeta]=\frac{1}{j\pi\omega\mu}\left(\mathcal{U}+k^2\mathcal{U}\right)+\frac{1}{j\omega\mu}\left[\left(P_i''+k^2P_i\right)+\left(P_e+k^2P_e\right)\right]$$

To solve this equation we expand the solution in terms of powers of $x = \frac{1}{\Omega}$. By our assumption of a very narrow slot we have $kd = \frac{2\pi d}{\lambda} < 1$, so $\ell n \frac{2\pi d}{\lambda}$ is a large number. Hence x is a small number, and expanding the voltage we write

$$V(\zeta)=V_0(\zeta)+V_1(\zeta)x+V_2(\zeta)x^2+\cdots \tag{2.35}$$

with the boundary condition that $V_i(\pm \ell) = 0$. In general, it will be sufficient for our approximation to take the first term that is different from zero. Substituting (2.35) in (2.34) we get

$$V_0'' + k^2 V_0 = 0 \qquad V_0(\pm \ell) = 0$$

$$V_1'' + k^2 V_1 = \frac{j\omega\mu\pi}{2}\left\{H_3^0 + \mathcal{K}[V_0(\zeta), \zeta]\right\} \qquad V_1(\pm \ell) = 0$$

$$V_2'' + k^2 V_2 = \frac{j\omega\mu\pi}{2}\mathcal{K}[V_1(\zeta), \zeta] \qquad V_2(\pm \ell) = 0$$

(2.36)

This gives us a set of differential equations, with boundary conditions, for the determination of successive approximations for the induced voltage. The values we see are functions of the unperturbed exciting field H^0 and the function $\mathcal{K}(V_i, \zeta)$. This is a rather complicated function of both the voltage and the geometry of the slot. Later we shall investigate this function in detail.

A SLOT OF LENGTH $2\ell = \frac{n\lambda}{2}$

For such a slot we find that the solution for V_0 which will satisfy the boundary conditions is

$$V_0(\zeta) = C_0\varphi(\zeta) = C_0 \begin{cases} \sin k\zeta & \text{for } n \text{ even} \\ \cos k\zeta & \text{for } n \text{ odd} \end{cases} \qquad (2.37)$$

It is to be noted that in this case of a resonant slot, the functional variation of the dominant component of the voltage will be sinusoidal across the slot, regardless of the form or composition of the exciting field H^{o}.

We note however, that the first equation in (2.36) does not tell us anything about the amplitude, which is a major thing we are attempting to find. However, to get the amplitude we can utilize the particular boundary conditions of this perturbation method. Let us multiply the left side of the second equation of the system (2.36) by V_o and integrate from $-\ell$ to $+\ell$. We have

$$Q = \int_{-\ell}^{\ell} (V_o V_1'' + k^2 V_o V_1)\, d\xi = \int_{-\ell}^{\ell} (V_o V_1'' - V_o'' V_1)\, d\xi$$

Integrating both terms by parts we get

$$Q = V_o V_1' \Big|_{-\ell}^{+\ell} - \int_{-\ell}^{\ell} V_1' V_o'\, d\xi - \Big[V_o' V_1 \Big|_{-\ell}^{\ell} - \int_{-\ell}^{\ell} V_o' V_1'\, d\xi \Big]$$

Noting now that

$$V_o(\pm\ell) = 0 \qquad \text{and} \qquad V_1(\pm\ell) = 0$$

by boundary conditions, we find that $Q = 0$

Let us then multiply the right side of the second equation in (2.36) by V_o and integrate over the interval $[-\ell, +\ell]$ with the result that

$$\int_{-\ell}^{\ell} \{ H_z^{o} + \mathcal{K}[V_o(\xi), \xi] \} C_o \varphi(\xi)\, d\xi = 0 \tag{2.38}$$

As $\mathcal{K}[V_o(\xi), \xi]$ is a homogeneous function of 1st degree in $V_o(\xi)$ we have

$$\mathcal{K}[V_o(\xi), \xi] = C_o\, \mathcal{K}[\varphi(\xi), \xi]$$

and we get from (2.38)

$$C_0 = -\frac{\int_{-\ell}^{\ell} H_z^{\circ}\,\varphi(\xi)\,d\xi}{\int_{-\ell}^{\ell} K[\varphi(\xi),\xi]\cdot\varphi(\xi)\,d\xi} = \frac{\int_{-\ell}^{\ell} H_z^{\circ}\,\varphi(\xi)\,d\xi}{-Y_m} \tag{2.39}$$

Hence we see how to determine the amplitude of the excited voltage V_0 as a function of H^0 and Y_n.

$$V_0(\xi) = \frac{\int_{-\ell}^{\ell} H_z^{\circ}\,\varphi(\xi)\,d\xi}{-Y_m}\cdot\varphi(\xi) \tag{2.40}$$

SLOT OF ARBITRARY LENGTH

In this case it is obvious that the equation for V_0 in (2.36) with the boundary condition $V_0(\pm\ell) = 0$ has only the trivial solution $V_0(\xi) = 0$. The first term in (2.35) different from zero will be $V_1(\xi)$. From the second equation in (2.36), where we note that $K(V_0,\xi) = 0$ as K is linear, we find

$$V_1(\xi) = C_1 \sin k(\xi+\ell) + \frac{j\omega\mu\pi}{2k}\int_0^{\xi+\ell} H^{\circ}(x)\sin(\xi+\ell-x)\,dx \tag{2.41}$$

where

$$C_1 = \frac{-j\omega\mu\pi}{2k\sin 2k\ell}\int_0^{2\ell} H^{\circ}(x)\sin k(2\ell-x)\,dx \tag{2.42}$$

Therefore, for a slot of arbitrary length both the amplitude and distribution of the induced voltage are dependant on the exciting field. This shows that the usual assumption that the distribution over the slot is purely sinusoidal is valid only for resonant or almost resonant slots ($2\ell \approx \frac{m\lambda}{2}$). In the general case this assumption is no longer true.

In the coming investigation of the scattering by a slot, we shall limit ourselves to resonant slots. In the multimode guide we shall also have to consider briefly slight changes in resonant slot length, that is, almost resonant slots. As is seen from (2.40) to determine the amplitude C_o we have to know the function Y_n. The evaluation of this function is a major part of this theory, and an approximate evaluation will be described in Chapter IV.

CHAPTER III

SCATTERING MATRIX OF THE SLOT

Having determined the induced voltage in the slot, Chapter II, we can solve now the second part of the boundary value problem; that is, given the voltage over the slot, find the scattered field set up by it everywhere in the waveguide. Let us consider a resonant slot $2\ell = \frac{n\lambda}{2}$. To compute the scattering coefficients we apply again the Lorentz theorem. The method is the same as the one presented in S. Silver's book(6). The expressions are rewritten here simply for convenience, and to make the derivation of the scattering coefficients complete.

Consider a waveguide with a slot along its generating line.

In this waveguide we separate out a region of space bounded by the walls (assumed to be perfectly conducting) and two cross sections at $z = z_2$, far enough from $z = 0$ so that only the propagating modes are of a significant amplitude, (see Fig. 6). This is a source-free region and we have the following: given two fields of the same frequency (E,H) and (E',H') that satisfy Maxwells equation,

$$\iint_\sigma (\underline{E} \times \underline{H}' - \underline{E}' \times \underline{H}) \cdot \underline{n}\, d\sigma = \iint_\sigma \underline{T} \cdot \underline{n}\, d\sigma = 0 \tag{3.1}$$

Since both E and E' are solutions of Maxwell's equations, they satisfy the boundary conditions on the walls.

$$n \times E = 0 \; ; \; n \times E' = 0$$

therefore, the integration over the walls will not contribute to the integral in (3.1). We are left with an integration over the slot and the two cross sections at $z = z_1$ and $z = z_2$.

That is,

$$\int_{SLOT} \underline{T} \cdot \underline{n}\, d\sigma + \int_{z=z_1} \underline{T} \cdot \underline{n}\, d\sigma + \int_{z=z_2} \underline{T} \cdot \underline{n}\, d\sigma = 0 \tag{3.2}$$

We choose now for the field (E,H), the combined field in the waveguide due to the incident wave and the scattered waves due to the slot. For the field (E',H'), we choose a particular unperturbed characteristic mode that will give us the desired scattering coefficient.

For the integral over the cross section at $z = z_1$ in (3.2) we have for TE modes

$$I_{z_1} = \int_0^a \int_0^b \left(-E_y H_x' + E_y' H_x\right) dx\, dy \tag{3.3}$$

Suppose now that the incident field is one particular mode of unity amplitude, and that $S_{i\nu}$ is the scattering coefficient of the slot. $S_{i\nu}$ gives the scattering into mode ν due to mode i incident. From the definitions in Chapter I we can write for the components of (E',H') at $z = z_1$ the following expressions

$$E_y' = \frac{-2j\omega\mu}{\sqrt{c\mu\beta_n(k^2-\beta_n^2)abk}} \cos\frac{n\pi x}{a}\, e^{-j\beta_n z}$$

$$H_x' = \frac{2j\beta_n}{\sqrt{c\mu\beta_n(k^2-\beta_n^2)abk}} \cos\frac{n\pi x}{a}\, e^{-j\beta_n z} \tag{3.4}$$

These expressions give the TE_{no} mode with the proper normalization as indicated in (1.5) and (1.6). For simplicity let us define a quantity $p_n(x)$ so that

$$p_n(x) = \frac{2}{\sqrt{c\mu k\beta_n(k^2-\beta_n^2)ab}} \cos\frac{n\pi x}{a} \tag{3.5}$$

and rewrite (3.4) in the form

$$
\begin{aligned}
E_y' &= -j\omega\mu\, P_m(x)\, e^{-j\beta_m z_1} \\
H_x' &= j\beta_m P_m(x)\, e^{-j\beta_m z_1}
\end{aligned}
\tag{3.6}
$$

With this notation, and in view of (1.9), we will have for the field (E,H) the following

$$
\begin{aligned}
E_y &= -j\omega\mu\, P_i(x) e^{-j\beta_i z_1} + \sum_{\nu=1}^{N} S_{i\nu} (-j\omega\mu) P_\nu(x) e^{j\beta_\nu z_1} \\
H_x &= j\beta_i P_i(x) e^{-j\beta_i z_1} + \sum_{\nu=1}^{N} S_{i\nu} - j\beta_\nu P_\nu(x) e^{j\beta_\nu z_1}
\end{aligned}
\tag{3.7}
$$

Substituting these expressions in (3.3) and carrying out the algebra, remembering the orthogonality of the modes[6], we get

$$
I_{z_1} = \frac{4\beta_m}{(k^2-\beta_m^2)\beta_m} \left(\frac{m\pi}{a}\right)^2 S_{im} \tag{3.8}
$$

and as $k^2 - \beta_m^2 = \left(\frac{m\pi}{a}\right)^2$ we find

$$
I_{z_1} = 4 S_{im} \tag{3.9}
$$

If we carry through the same computation at $z = z_2$ we will find that the integral I_{z_2} will be zero, as all the waves propagate in the same direction.

We have to compute now the value of the integral over the slot. Here we have

$$I_s = \int_{-d}^{d}\int_{-\ell}^{\ell}\left[(E_x H_z' - E_z H_x') + (E_x' H_z - E_z' H_x)\right] dx\,dz \tag{3.10}$$

By the boundary conditions on the unperturbed auxiliary field, we know that $E_z' = 0$ and $E_x' = 0$. By our approximation for a narrow slot we can neglect E_z in comparison with E_x and we get

$$I_s = -\int_{-d}^{d}\int_{-\ell}^{\ell} E_x (k^2 - \beta_m^2)\, \rho_m(x) e^{-j\beta_m z}\, dx\,dz \tag{3.11}$$

On the basis of the assumption of a narrow slot we can neglect the variation in $\rho_m(x)$ over the width of the slot and write

$$I_s = -(k^2 - \beta_m^2)\, \rho_m(x) \int_{-\ell}^{\ell} V(z) e^{-j\beta_m z}\, dz \tag{3.12}$$

where

$$V(z) = \int_{-d}^{d} E_x(x, z)\, dx \tag{3.13}$$

In Chapter II we have seen that, for a slot of a length equal to half free space wavelength, we have for the voltage across the slot, from (2.40)

$$V(z) = C_0^{\,i} \cos kz \tag{3.14}$$

where the form $\cos kz$ is independent of the particular configuration of the exciting field. The amplitude, on the other hand, is a function of the exciting field and is given by equation (2.39). Hence

$$C_o^i = \frac{\int_{-\ell}^{\ell} H_z^i \cos kz \, dz}{Y_m} \tag{3.15}$$

Substituting H_z^i in (3.15) we have

$$C_o^i = \frac{(k^2-\beta_i^2)}{Y_m} \int_{-\ell}^{\ell} \rho_i(x) e^{-j\beta_i z} \cos kz \, dz$$

as $\ell = \lambda/4$ we have $k\ell = \pi/2$ and we get

$$C_o^i = \frac{2k}{Y_m} \rho_i(x) \cos \beta_i \ell \tag{3.15a}$$

Substituting (3.14) in (3.12) we find

$$I_s = -(k^2-\beta_m^2)\rho_m(x) \int_{-\ell}^{\ell} C_o^i \cos kz \, e^{-j\beta_m z} dz \tag{3.16}$$

Now by (3.2)

$$I_{z_1} + I_s = 0 \tag{3.17}$$

and substituting (3.9) and (3.16) in (3.17) we find

$$4S_{im} = (k^2-\beta_m^2)\rho_m(x) \int_{-\ell}^{\ell} C_o^i \cos kz \, e^{-j\beta_m z} dz \tag{3.18}$$

$$Z_{ik} = \sum_{\nu=1}^{N} [(1) - (S)]^{-1}_{i\nu} [(1)+(S)]_{\nu k} \tag{3.22}$$

where

$$[(1)-(S)]^{-1}_{i\nu} = \frac{\{(1)-(S)\}_{i\nu}}{D[(1)-(S)]} \tag{3.23}$$

In (3.23) $D[(1)-(S)]$ is the determinant of the matrix $[(1)-(S)]$ and $[(1)-(S)]_{i\nu}$ is the cofactor[19] of the element $(i\nu)$ of $[(1)-(S)]$

We see that the impedance in (3.22) have a well defined meaning only if $[(1)-(S)]$ is a non-singular matrix. The case where $[(1)-(S)]$ is singular would correspond to the conditions of self oscillations in the mode coupling structure.

In expression (3.21) all terms buy y_n are real. Therefore, the phase of the scattering elements will be determined by the phase of y_n. Our next step in the theory will be the evaluation of y_n.

CHAPTER IV

EVALUATION OF THE ADMITTANCE FUNCTION

In the preceding three chapters we have outlined the solution of the boundary value problem in its two parts. We have derived an expression for the induced sources eq. (2.40) and the scattering coefficient of a slot eq. (3.21). In both expressions the function $Y_m[\varphi(\varsigma),\varsigma]$ that is defined in (2.39) enters. For a solution in closed form, directly applicable to numerical computation and practical design, we have to evaluate this function. We rewirte (2.39) for

$$Y_m = -\int_{-\ell}^{\ell} \mathcal{K}[\varphi(\varsigma),\varsigma]\cdot\varphi(\varsigma)\,d\varsigma \tag{4.1}$$

where (see eq. (2.34')

$$\mathcal{K}[\varphi(\varsigma),\varsigma] = \frac{1}{j\pi\omega\mu}(\mathcal{U}''+k^2\mathcal{U}) + \frac{1}{j\omega\mu}\left[(P_i''+k^2P_i)+(P_e''+k^2P_e)\right] \tag{4.2}$$

Let us first recognize the nature of the function Y_n. By (2.39) we have for the amplitude of the induced voltage as a function of the exciting field H^o the expression

$$C_o = -\frac{1}{Y_m}\int_{-\ell}^{\ell} H_t^{\circ}(\eta,y,\varsigma)\,\varphi(\varsigma)\,d\varsigma \tag{4.3}$$

We can represent the tangential magnetic field by an equivalent electric current. In order to demonstrate the nature of Y_n it is convenient to consider for a moment that the slot is center fed by a Dirac delta function source. From general network theory (Duhamel's Integral) we know that the response to this kind of driving current can be defined as an input admittance.

We apply then, at the center of the slot, a driving field of the form

$$H^{0}(\varsigma) = I_0\,\delta(\varsigma - \varsigma_0) \tag{4.4}$$

Substituting (4.4) in (4.5) we find that the amplitude of the induced voltage is

$$C_0 = \frac{I_0}{-Y_m} \tag{4.5}$$

We see, therefore, that in terms of circuit analogies the function $-Y_m$ takes the role of an admittance function. It is important to note that as $\frac{1}{-Y_m}$ would then be an impedance function, and $R_e(\frac{1}{-Y_m})$ would be the radiation resistance of the slot.

In evaluating this function let us divide the treatment into three parts. We shall evaluate each one of them separately. We write

$$-Y_m = Y_1 + Y_2 + Y_3 \tag{4.6}$$

where by (4.1) and (4.2)

$$Y_1 = \frac{1}{j\pi\omega\mu}\int_{-\ell}^{\ell}(U'' + k^2 U)\,\varphi(\varsigma)\,d\varsigma \tag{4.7}$$

and

$$Y_2 = \frac{1}{j\omega\mu}\int_{-\ell}^{\ell}(P_i'' + k^2 P_i)\,\varphi(\varsigma)\,d\varsigma \tag{4.8}$$

and

$$Y_3 = \frac{1}{j\omega\mu} \int_{-\ell}^{\ell} (P_e'' + k^2 P_e)\,\varphi(\zeta)\,d\zeta \tag{4.9}$$

These functions depend on the geometry of the slot and the geometry of the waveguide. Until now we have not restricted ourselves in order to leave the theory in as general a form as possible. To evaluate these functions explicitly we have to specify these factors.

We consider a longitudinal slot of length 2ℓ that is equal to $\frac{\lambda}{2}$ where λ is the free space wavelength of the exciting field. Further, we consider a rectangular guide that can propagate only TE_{no} modes, and the slot is cut in a broad face of the guide (see Fig. 6), parallel to the axis of the guide. With these assumptions the coordinate ζ will coincide with the coordinate z of the guide. If we put the center of the coordinate system xyz so that the center of the slot is at $z = 0$, we need not distinguish between ζ and z anymore. It is to be noted that these assumptions play a role mainly in computing Y_2 and Y_3. As we shall see, Y_1 is independent of guide geometry. This is to be expected as it is determined by the unperturbed Green's function of free space, (see Chapter II).

Let us first consider the function Y_1. From Chapter II we have for U_z

$$\mathcal{U}[\varphi(\zeta),\zeta] = \left(\int_{-\ell}^{z} - \int_{z}^{\ell}\right) \ln k\left(k + \sqrt{k^2 - m^2}\right)\left[\varphi'(z') - jk\varphi(z'')\ \frac{z' - z}{R}\right] e^{-jkR}\,dz' \tag{4.10}$$

We make now some simplifying transformations. Let us re-state the assumption that the slot is narrow. With an error of order $kd \ll 1$ we can write from (2.19)

$$R = |z - z'| \tag{4.11}$$

This is equivalent to saying that we shall integrate along the center line of the slot. In doing so we are bound to pass a singularity point at $z = z'$. For this reason we had to take the integral in (2.17) in the principal part sense. With the assumption (4.11) we can write

$$k(l+\sqrt{R^2-m^2}) \approx 2k|z-z'| \tag{4.12}$$

Substituting (4.12) in (4.10) and noting the signs we get

$$\mathcal{U}[\varphi(z)z] = \int_{-l}^{z} \ln 2k(z-z')(\varphi'+jk\varphi)e^{-jk(z-z')}dz' - \int_{z}^{l} \ln 2k(z'-z)(\varphi'-jk\varphi)e^{-jk(z'-z)}dz' \tag{4.13}$$

We have to compute $\frac{\partial^2 \mathcal{U}}{\partial z^2}+k^2\mathcal{U}$; however, we note that by the seperation in (4.13) both integrals are regular, and we can interchange differentiation with integration. Also note that we differentiate with respect to z and integrate with respect to z'. Carrying out the differentiation and remembering that by boundary conditions we have $\varphi(\pm l)=0$ we get a convenient expression for $\mathcal{U}[\varphi(z)z]$ *. Noting further that in our case we have $\varphi''(\pm l)=0$, and that $\varphi''+k^2\varphi=0$, $\varphi'''+k^2\varphi'=0$, we finally get the rather simple form

$$\mathcal{U}''+k^2\mathcal{U} = \frac{ke^{-jk(l+z)}}{l+z} + \frac{ke^{-jk(l-z)}}{l-z} \tag{4.13}$$

Substituting (4.14) in (4.7) we get

*Particulars on this transformation are given in appendix (2).

$$Y_1 = \frac{1}{\pi} q \int_{-\ell}^{\ell} k \cos kz \left[\frac{e^{-jk(\ell+z)}}{\ell+z} + \frac{e^{-jk(\ell-z)}}{\ell-z} \right] dz \tag{4.15}$$

where

$$q = \frac{1}{j\omega\mu}$$

In our case $2\ell = \frac{\lambda}{2}$ we have $k\ell = \frac{\pi}{2}$. We can write therefore

$$\cos kz = \sin k(\ell + z) = \sin k(\ell - z)$$

Changing variables, by substituting $k(\ell + z) = t$ in the first part of (4.15) and $k(\ell - z) = t$ in the second part of (4.14), we can rewrite Y_1 as follows

$$Y_1 = \frac{q}{\pi} \int_0^{2\ell} 2k \frac{\sin t}{t} e^{-jt} dt \tag{4.16}$$

but

$$\sin t e^{-jt} = \sin t \cos t - j \sin^2 t = \frac{1}{2}\left[\sin 2t - j(1 - \cos 2t)\right]$$

substituting now $2t = x$ we find

$$Y_1 = G_1 + jB_1 = \frac{1}{c\mu\pi} D_i(2\pi) - j\frac{1}{c\mu\pi} S_i(2\pi) \tag{4.17}$$

From (4.16) and (4.17) we find, substituting the value of q and $k = \frac{\omega}{c}$ (c velocity of light),

$$Y_1 = G_1 + j B_1 = -\frac{1}{c\mu\pi} D_i(2\pi) - j\frac{1}{c\mu\pi} S_i\,(2\pi) \tag{4.18}$$

where $S_i(x)$ and $D_i(x)$ are the sine and cosine integrals, that is

$$S_i\,(x) = \int_0^x \frac{\sin t}{t}\,dt$$

and

$$D_i\,(x) = \int_0^x \frac{1-\cos t}{t}\,dt$$

Considering only the resistive part we can see that Y_1 corresponds to the result usually employed for the admittance of a $\frac{\lambda}{2}$ slot(6,9). The value employed is the external impedance of a slot in a perfectly conducting plane. For this part of the radiation resistance of the slot we have from (4.18)

$$R_s = \frac{1}{G} = \frac{c\mu\pi}{D_i\,(2\pi)} \tag{4.19}$$

Now, for a half wave complementary wire antenna we have[15]

$$R_d = \sqrt{\frac{\mu}{\epsilon}}\,\frac{1}{4\pi} D_i\,(2\pi) = 73.129\,\Omega \tag{4.20}$$

Comparing (4.20) and (4.19) it is evident that

$$\rho_s = \frac{1}{4R_d}\,\frac{\mu}{\varepsilon} \tag{4.21}$$

We see now that Y_1 gives us a value corresponding to the slot as an analog of its complementary wire antenna. It can be considered as the first approximation in our theory. The other terms Y_2 and Y_3 of (4.6) will give us correction terms to add to this first order approximation. Let us now consider the term Y_2.

From (4.8) we see that to determine Y_2 we have to find the function $P_i[\varphi(\zeta)\zeta]$. By the discussion in Chapter II and equations (2.26) and 2.13) we have

$$P_i\left[\varphi(\zeta)\zeta\right] = \int_{-l}^{l} \varphi(\zeta')\, g_i(\eta\zeta, \eta\zeta')\, d\zeta' \tag{4.22}$$

where the function $g(\eta\zeta, \eta'\zeta')$ is the one introduced in (2.13). As indicated before, finding an accurage expression for g_i would be a very difficult task in itself. We are interested in Y_2 as a correction to the basic term Y_1, and one which will enable us to take account of the difference between the radiation fields in the interior and exterior regions. This difference was pointed out before, and one should expect it to be of importance in the phenomena. With this in mind we look for an approximate expression for $g_i(z,z')$.

To find an approximate expression for g_i let us note that we can easily find the Green's functions for the inside of a cylindrical waveguide in terms of an infinite series. We can write for the Green's function[3]

$$G(xyz, x'y'z') = \sum_{m=1}^{\infty} \frac{\alpha_m^2\, \psi_m(x,y)\, \psi_m(x'y')}{-2\gamma_m} e^{\gamma_m |z-z'|} \tag{4.23}$$

The functions $\psi_n(x,y)$ are the eigengunctions of the equation

$$\nabla^2\psi + K\psi = 0 \tag{4.24}$$

for the particular cross-section of the guide, with the boundary conditions

$$\frac{\partial \psi_m(xy)}{\partial \underline{n}} = 0 \qquad \text{or} \qquad \psi_m(xy) = 0 \tag{4.25}$$

corresponding to the case of TE or TM modes. The γ_m are the propagation constants and α_m the normalization constants of the mode function $\psi(x,y)$ defined so that

$$\iint [\alpha \psi_m(x,y)]^2 d\sigma = 1 \tag{4.26}$$

comparing now (4.23) and (2.13) we write

$$\sum_{m=1}^{\infty} \frac{\alpha_m^2 \psi_m(xy)\psi_m(x'y')}{-2\gamma_m} e^{\gamma_m|z-z'|} = \frac{1}{2\pi}\frac{e^{-jkR}}{R} + g_i(\rho,\rho') \tag{4.27}$$

It is well known that the γ_m for all modes from 1 to N, where N is a finite number depending on the cross section of the guide, will be imaginary. For all modes of index $n > N$ the γn are real, and they represent non-propagating modes. Let us now break up the sum in (4.23 into two, and write

$$G(\rho,\rho') = \sum_{m=1}^{N} \frac{\alpha_m^2 \psi_m(xy)\psi_m(x'y')}{-2\gamma_m} e^{\gamma_m|z-z'|} +$$

$$\sum_{m=N+1}^{\infty} \frac{\alpha_m^2 \psi_m(x,y)\psi_m(x'y')}{-2\gamma_m} e^{\gamma_m|z-z'|} \tag{4.28}$$

Consider now the asymptotic form of $G(P,P')$ as the observation point goes to infinity.

$$\lim_{R\to\infty} G(P,P') = \lim_{z-z'\to\infty}\left[\sum_{m=1}^{N} \frac{\alpha_m^2 \psi_m(x,y)\,\psi_m(x',y')}{-2j\beta_m} e^{j\beta_m|z-z'|}\right.$$

$$\left. + \sum_{m=N+1}^{\infty} \frac{\alpha_m^2 \psi_m(x,y)\,\psi_m(x',y')}{-2\gamma_m} e^{\gamma_m|z-z'|}\right] \qquad (4.29)$$

In the second summation γ_m is real and therefore as $z - z'$ goes up all the terms go to zero. We have, therefore, that far away from the source point z' the Green's function becomes

$$G_{\pm\infty} = \sum_{m=1}^{N} \frac{\alpha_m^2 \psi_m(x,y)\,\psi_m(x',y')}{2j\beta_m} e^{j\beta_m|z-z'|} \qquad (4.30)$$

where N is the number of propagating modes. Further, let us see what the form of $G(P,P')$ is for large distances, in terms of the second representations. We have

$$\lim_{R\to\infty} G(P,P') = \lim_{R\to\infty} \frac{e^{-jkR}}{R} + \lim_{R\to\infty} g_i(P,P') \qquad (4.31)$$

and as e^{-jkR} is finite (k is real) the first term goes to zero. Hence we get, therefore, that

$$G_{\pm\infty}(P,P') = g_{i\,\pm\infty}(P,P')$$

comparing (4.32) with (4.30) we see that for large distances

$$g_{i}_{t\infty} = \sum_{m=1}^{N} \frac{\alpha_m^2 \, \psi_m(xy) \, \psi_m(x'y')}{-2j\beta_m} e^{j\beta_m |z-z'|} \tag{4.33}$$

It is seen now how we can find the asymptotic values of g_i for large distances. It should also be noted that this expression (4.33) depends on the propagating unattenuated modes. Referring back to the discussion in the introduction of the question of radiation conditions and far zone fields, we see that this form brings out this very difference. As we are looking for an approximation, the simplest assumption would be to take this far zone asymptotic form. This is an assumption, and it gives us an approximation to the true Green's function. It is a plausible approximation, and the results support this point of view.

It may be worth pointing out here that the $g_i(P,P')$ as chosen makes $G(P,P')$ satisfy condition (2.10a) (2.10b) and (2.10d). If it would have made $G(P,P')$ satisfy (2.10c) too, everywhere it would be the rigorous $g_i(P,P')$. However, it makes $G(P,P')$ satisfy (2.10c) only on the wall in which the slot is cut, and everwhere at a distance from the slot of the order of several wavelengths. Although the order of error involved is not shown mathematically, on the basis of these arguments, it is believed to be appreciably smaller than the correction represented by the $g_i(P,P')$ which is used.

On the basis of this g_i we can now find $P_i[\varphi(z),z]$. From (4.22), noting that for a narrow slot $\psi(x,y)$ and $\psi(x'y')$ are essentially the same, we have

$$P_i[\varphi(z),z] = \sum_{m=1}^{N} \frac{\alpha_m^2}{-2j\beta_m} \psi_m^2(xy) \int_{-l}^{l} \varphi(z') e^{j\beta_m |z-z'|} dz \tag{4.35}$$

To find Y_2 we have to compute $P_i'' + K^2 P_i$. We have to differentiate this expression twice with respect to z (integration with respect to z'). The only part depending on z is the integral in (4.35). Carrying out this computation* we find

---

*Particulars on this transformation see in appendix (3).

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\int_{-\ell}^{\ell} e^{j\beta|z-z'|}\varphi(z')\,dz' = k\left[e^{j\beta(\ell-z)} + e^{j\beta(\ell+z)}\right] \tag{4.36}$$

Having computed this we can now find the value of Y_2 to within this approximation. By (4.8)

$$Y_2 = \frac{1}{j\omega\mu}\int_{-\ell}^{\ell}\left(\sum_{m=1}^{N} \frac{\alpha_m^2}{-2j\beta_m}\,\psi_m^2\, Re^{j\beta_m \ell}\, 2\cos\beta_m z\right)\cdot\varphi(z)\,dz \tag{4.37}$$

Hence we have

$$Y_2 = \sum_{m=1}^{N} \frac{g\alpha_m^2 k}{-j\beta_m}\,\psi_m^2(xy)\,e^{j\beta_m \ell}\int_{-\ell}^{\ell}\cos\beta_m z\,\cos kz\,dz \tag{4.38}$$

From this, after some simple algebraic transformation, we find

$$Y_2 = \sum_{m=1}^{N} \frac{g\alpha_m^2\,\psi_m^2(xy)\,k^2}{j\beta_m(k^2-\beta_m^2)}\left(\cos^2\beta_m\ell + \frac{1}{2}j\sin 2\beta_m\ell\right) \tag{4.39}$$

As we have considered the case of a rectangular waveguide of sides 'a' and 'b', and such that only TE_{no} modes propagate we have

$$\psi_m(x,y) = \cos\frac{m\pi x}{a}\ ;\ \alpha_m = \sqrt{\frac{2}{ab}} \tag{4.40}$$

Substituting this, and $g = \frac{1}{j\omega\mu}$ and $k = \frac{\omega}{c}$, we find finally for Y_2

$$Y_2 = G_2 + jB_2 = \frac{1}{c\mu\pi}\,\frac{4\pi}{ab}\sum_{m=1}^{N}\frac{k\cos^2\frac{m\pi x}{a}}{\beta_m(k^2-\beta_m^2)}\left(\cos^2\beta_m l+\frac{1}{2}\sin 2\beta_m l\right) \qquad (4.41)$$

This Y_2 can be considered as the internal admittance of the slot. It indeed gives the response of the slot in its relation to the propagating modes.

As we pointed out before, Y_2 gives a correction term for the admittance. We note from (4.41) that it depends on the coordinate x through $\cos^2\frac{n\pi x}{a}$. That tells us that as the slot is moved away from the wall $x = \begin{cases} 0 \\ a \end{cases}$, the correction term decreases and finally at $x = \frac{a}{2}$ disappears. This is to be expected as it is well known that the coupling of the slot to the propagating TE_{n0} mode goes down as the slot approaches the center. At the center the slot effectively does not see the far zone field in the guide.

Having computed Y_2 approximately, let us now find Y_3 to the same order of approximation. From (4.9) we see that to find Y_3 we must know the function P_e. By (2.29) and (2.30) we have

$$P_e[\varphi(z), z] = \int_{-\ell}^{\ell} \varphi(z') g_e(xyz, x'y'z') dz' \qquad (4.42)$$

where $g_e(P,P')$ is defined by (2.28). For the outside region we have

$$P_e[\varphi(z), z] = \int_{-\ell}^{\ell} \varphi(z') g_e(xyz, x'y'z') dz' \qquad (4.43)$$

It would not be easy to write another expansion for the Green's function $G_e(P, P')$ for the outside space, that will satisfy the proper boundary conditions on the walls of the rectangular waveguide outside. However, if we say that $G_e(P,P')$ is the correct function that presumably can be found, we have at our disposal the knowledge of its asymptotic behavior at infinity. By the Sommerfeld radiation conditions we know that as R goes to infinity $G_e(P,P')$ must go to zero at least as fast as $\frac{1}{R}$. Hence from (4.43) we see that

$$\lim_{R \to \infty} g_e(P,P') = 0 \qquad (4.43)$$

as k is real and e^{-jkR} is finite.

If we are to be satisfied with the same order of approximation as used for the inside of the waveguide, we would take $g_e(P,P')$ everywhere equal to its asymptotic value as $R \to \infty$. We get therefore that within this approximation $g_e(P,P') = 0$ and $P_e[\varphi(z), z] = 0$ as well. As a result we conclude that within our approximation

$$Y_3 = 0 \qquad (4.44)$$

It should be pointed out here that if we assumed an infinite plane outside the waveguide, the correct Green's function for the outside region would be

$$G_e(P,P') = \frac{1}{2\pi} \quad \frac{e^{-jkR}}{R}$$

Therefore $g_e(P,P')$ would rigorously be equal to zero. This means that with such an assumption $Y_3 = 0$ rigorously. Hence we see that assuming an infinite plane outside should yield the same results as obtained within our approximation. From this discussion it is also evident that adding finite wings (plane) to the outside walls would make the approximation better [illegible], since the assumption that $g_e(P,P')$ is zero becomes closer to the physical condition. In particular, a difference would be observed if the slot were near the edge of the waveguide wall, as the change in geometry occurs near the source point. This fact has been observed by us experimentally. We found little difference as long as the slot was not at the very edge of the wall. For slots at the edge there was a noticeable difference in the scattering as measured inside. For experimental reasons we had to have the slots at the edge of the wall, and therefore decided to add small wings to the wall of the waveguide as a plane conductor.

From (4.6) we can now, by adding (4.18), (4.41) and (4.44), write the function Y_n in closed form.

$$Y_n = G + jB \tag{4.45}$$

where

$$G = \frac{1}{c\mu\pi}\left[D_i(2\pi) - \frac{4\pi}{ab}\sum_{m=1}^{N} \frac{k \cos^2 \frac{m\pi x}{a}}{\beta_m(k^2-\beta_m^2)} \cos^2 \frac{\pi}{2}\frac{\beta_m}{k}\right] \tag{4.46}$$

and

$$B = \frac{1}{c\mu\pi}\left[S_i(2\pi) - \frac{2\pi}{ab}\sum_{m=1}^{N} \frac{k \cos^2 \frac{m\pi x}{a}}{\beta_m(k^2-\beta_m^2)} \sin \pi \frac{\beta_m}{k}\right] \tag{4.47}$$

In (4.46) and (4.47) we substituted $\ell = \frac{\pi}{2k}$. These expressions are easily evaluated as they involve only a finite summation. In a later chapter we shall see the numerical values in a double mode guide.

It should also be pointed out here that the function Y_n for a half wave slot within our approximation is independent of the width of the slot for relatively narrow slots (see Chapter II). This is ture within this approximation for a half wave slot only. As can be seen from (2.41) and (2.35) for a slot of arbitrary length the induced voltage will be proportional to $\chi = \ln kd$. If one evaluated the function Y_n for a slot of arbitrary length one would find that Y_n too would depend on the width d. The fact that the function Y_n for a half wave slot is insensitive to the width is well demonstrated by our experimental results[17,18].

CHAPTER V

SCATTERING IN A DOUBLE MODE GUIDE

If we substitute expression (4.46) and (4.47) in (3.21) we obtain an expression for the scattering coefficient in closed form

$$S_{ik}=\frac{\frac{4\pi}{ab}\frac{k}{\sqrt{\beta_i\beta_k(k^2-\beta_i^2)(k^2-\beta_k^2)}}\cos\frac{k\pi x}{a}\cos\frac{i\pi x}{a}\cos\beta_i l\cos\beta_k l}{\left[D_i(2\pi)-\frac{4\pi}{ab}\sum_{v=1}^{N}\frac{k\cos^2\frac{2\pi x}{a}}{\beta_v(k^2-\beta_v^2)}\cos^2\frac{\beta_v}{k}\frac{\pi}{2}\right]+j\left[S_i(2\pi)-\frac{2\pi}{ab}\sum_{v=1}^{N}\frac{k\cos^2\frac{v\pi x}{a}}{\beta_v(k^2-\beta_v^2)}\sin\frac{\beta_v}{k}\pi\right]} \tag{5.1}$$

This expression is readily computed. However, it would be worthwhile to compute the function Y_n separately for the case of a double mode guide. A rectangular guide that can propagate the TE_{10} and TE_{20} modes has been investigated experimentally[18]. Most of the experimental data have been taken from the work reported by this Laboratory before[17,18].

Let us consider the values of (4.46) and (4.47) first in a waveguide of dimensions $a = 4.064$ cm, $b = 1.016$ cm. The operating frequency is 9375.10^6 c.p.s., which corresponds to a free space wavelength of 3.2 cm. The length of the slots was $\frac{\lambda}{2} = 1.6$ cm. The slots of varying width were cut at different distances from the edge of the wall. It was found that the cross coupling between the modes is rather small. To get reliable data within the experimental errors of the system, it is necessary to take measurements with slots very close to the edge of the wall. This is the case for which we therefore propose to do the numerical work. For this case we have $x = 0$ or a, and $\cos\frac{2m\pi x}{a} = 1$. For a wavelength of 3.2 cm we have

$$k = \frac{2\pi}{\lambda} = 1.962 \ \text{cm}^{-1} \tag{5.2}$$

$$\beta_1 = \frac{2\pi}{\lambda_{g_1}} = 1.21 \ \text{cm}^{-1} \tag{5.3}$$

and

$$\beta_2 = \frac{2\pi}{\lambda_{g_2}} = 1.21 \ \text{cm}^{-1} \tag{5.4}$$

The sum in expression (4.46) and (4.47) has two terms, due to the TE_{10} and TE_{20} modes. Let us compute them separately for the sake of clarity.

The first component gives for the real part

$$\frac{4\pi k}{ab\beta_1(k^2-\beta_1^2)} \cos^2 \frac{\pi}{2}\frac{\beta_1}{k} = \frac{4\pi}{4.06\;1.016}\;\frac{1.962}{1.804\;0.596}\cos^2\left(\frac{1.803}{1.962}\,90^\circ\right)$$

and for the imaginary part

$$\frac{2\pi}{ab}\frac{k}{\beta_1(k^2-\beta_1^2)}\sin\pi\frac{\beta}{k} = \frac{2\pi}{4.06\;1.016}\;\frac{1.962}{1.804\;0.596}\sin\left(\frac{1.803}{1.962}\,180^\circ\right)$$

after the arithmetic we get

$$0{,}0866 + j0{,}710 \tag{5.5}$$

The second component gives for the real part

$$\frac{4\pi}{ab}\frac{k}{\beta_2(k^2-\beta_2^2)}\cos^2\frac{\pi}{2}\frac{\beta_2}{k} = \frac{4\pi}{4.06\cdot 1.016}\;\frac{1.962}{1.21\cdot 2.38}\cos^2\left(\frac{1.21}{1.962}\,90^\circ\right)$$

and for the imaginary part

$$\frac{2\pi}{4.06\cdot 1.016}\cdot\frac{1.962}{1.21\cdot 2.38}\sin\left(\frac{1.21}{1.962}\cdot 180^\circ\right)$$

This gives

$$0{,}664 + j0.979 \tag{5.6}$$

Hence we find for the sum in (4.46) the value of 0,752 and for the sum in (4.46) the value of 1,69. From tables for the sine and cosine integrals[20] we find that $D_i(2\pi) = 2.44$ and $S_i(2\pi) = 1.417$. Substituting these values in (4.46) and (4.47) we get

$$G = \frac{1}{c\mu\pi}(2.44 - 0.75) = \frac{1.69}{120\pi^2} \tag{5.7}$$

and

$$B = \frac{1}{c\mu\pi}(1.417 - 1.69) = \frac{0.273}{120\pi^2} \tag{5.8}$$

Hence

$$Y_m = \frac{1}{120\pi^2}(1.69 - j0.273) \tag{5.9}$$

As was pointed out in Chapter IV, this function represents the admittance of a center fed slot. Its inverse would then give a function representing an input impedance defined on the same basis

$$Z = \frac{1}{Y_m} = \frac{120\pi^2}{1.69 - j0.273} = 120\pi^2(0.576 + j0.093) \tag{5.10}$$

Numerically this gives us for Z

$Z = 681 + j110$ ohms

which indicates agreement with measured values for the radiation impedance of center fed slots[16].

It should be noted from (5.5) and (5.6) that the contribution to the correction factor is mainly due to the second mode. A close look at (4.46) and (4.47) will indicate that for an arbitrary number of modes only the last one will contribute appreciably to Y. This is due to the fact that the propagation constant of the lower modes are very close to k. The cosine and sine approach $\frac{\pi}{2}$ and π as $\beta \to k$. Only for the higher modes is β_2 appreciably different from k, and the correction terms are of importance. This already indicates that as far as the slot's behavior is concerned it effectively sees only the higher mode. This phenomena will be further demonstrated by the reflections from the slot.

By the expressions (3.21) and (5.10) we can now rewrite (5.1) in the form

$$S_{mm} = \frac{4\pi}{a \cdot b}(0.576 + j\,0.093)\ \frac{k \cos\frac{m\pi x}{a} \cos\frac{m\pi x}{a}}{\sqrt{\beta_m \beta_m (k^2-\beta_m^2)(k^2-\beta_m^2)}}\ \cos\left(\frac{\beta_m}{k}90^\circ\right)\cos\left(\frac{\beta_m}{k}90^\circ\right) \tag{5.11}$$

For slots at the edge of the wall $\cos\frac{m\pi x}{a} = 1$, and the dependance on the x coordinate drops out. Suppose now that a wave of mode TE_{10} is incident. The reflected wave in mode TE_{10} will be:

$$S_{11} = \frac{4\pi}{4.13}\ \frac{k(0.576 + j\,0.093)}{\beta_1 (k^2 - \beta_1^2)}\ \cos^2\left(\frac{\beta_1}{k}90^\circ\right)$$

$$S_{11} = \frac{4\pi}{4.13}\ \frac{1.962(0.576 + j\,0.093)}{1.803 \cdot 0.595}\ \cos^2\left(\frac{1.803}{1.962}90^\circ\right) \tag{5.12}$$

hence

$$S_{11} = 10^{-3}(49.8 + j\,8) \tag{5.13}$$

The reflected wave in mode TE_{20} will be

$$S_{12} = \frac{4\pi}{4.13} \frac{k(0.576+j0.093)}{\sqrt{\beta_1\beta_2(k^2-\beta_1^2)(k^2-\beta_2^2)}} \cos\left(\frac{\beta_1}{k} 90^\circ\right) \cos\left(\frac{\beta_2}{k} 90^\circ\right)$$

$$S_{12} = \frac{4\pi \cdot 1.962(0.576+j0.093)}{4.13\sqrt{1.803 \cdot 1.21 \cdot 0.595 \cdot 2.38}} \cdot 0.125 \cdot 0.565 \tag{5.14}$$

hence

$$S_{12} = 10^{-3}(138 + j22) \tag{5.15}$$

Let us suppose now that a wave of mode TE_{20} is incident on the slot. The reflected wave in mode TE_{10} is incident. This symmetry is evident from (5.11). Hence

$$S_{21} = 10^{-3}(138 + j22) \tag{5.15a}$$

The reflected wave in mode TE_{20} will be

$$S_{22} = \frac{4\pi}{a \cdot b} \frac{k(0.576+j0.093)}{\beta_2(k^2-\beta_2^2)} \cos^2\left(\frac{\beta_2}{k} 90^\circ\right)$$

(5.16)

$$S_{22} = \frac{4\pi}{4.13} \frac{1.962(0.576+j0.093)}{1.21 \cdot 2.38} 0.565$$

hence

$$S_{22} = 10^{-3}(382 + j62) \tag{5.17}$$

As the waveguide is matched in both directions the scattered field to the right is the same as the scattered field to the left. Therefore the above calculated coefficients are sufficient to determine the field everywhere.

To measure the amplitudes of the reflection coefficients we measure the power of the reflected waves. These are plotted in the graphs in terms of db. below incident power. For the sake of comparison with the measured values let us express the coefficient in terms of decibels. We get from (5.13), (5.15) and (5.17) the following values:

$$S_{11} = 20 \lg_{10} 50 \quad 10^{-3}$$

$$= -26.1 \text{ db.}$$

$$S_{12} = S_{21} = 20 \lg_{10} 139 \quad 10^{-3}$$

$$= -17.2 \text{ db.}$$

$$S_{22} = 20 \lg_{10} 385 \quad 10^{-3}$$

$$= -8.4 \text{ db.}$$

To compare with measured values let us write them in a table

	Calculated db.	Measured db.
S_{11}	-26.1	-27.4 ± 1.5
S_{12}	-17.2	-18.7 ± 1.5
S_{21}	-17.2	-19.0 ± 1.5
S_{22}	- 8.4	-10.2 ± 1.5

From these values we can write now for the matrix (S) of a half wavelength slot in a rectangular guide the following:

$$
(S)\,(-1)\begin{pmatrix} 0.05+j0.009 & 0.14+j0.02 & -0.95+j0.009 & 0.14+j0.002 \\ 0.14+j0.02 & 0.4+j0.06 & 0.14+j0.02 & -0.6+j0.06 \\ -0.95+j0.009 & 0.14+j0.02 & 0.05+j0.009 & 0.14+j0.0.02 \\ 0.14+j0.02 & -0.6+j0.06 & 0.14+j0.02 & 0.4+j0.06 \end{pmatrix}.
$$

We see therefore that the results are, within the experimental error, in very good agreement. For the phase of the reflected amplitudes we have

$$
\operatorname{Tang}^{-1}\frac{-0.093}{-0.576} = -170.3^{\circ}
$$

We find therefore, that the phase angle is -170. The measured values are $-159^{\circ} \pm 10^{\circ}$. The agreement in phase is not so good, but it is a characteristic affair in radiation theory. The phase is sensitive to small changes in resonant length. It would be of interest to see the change in the function Y_n for slight changes in length.

Slots Very Close To Resonant Length

Suppose that the length of the slot is slightly off the resonant length $\ell_0 = \frac{\pi}{2k}$. This may be due to machining errors, frequency errors, or possibly purposely introduced by the design. Let us denote the length by ℓ. Assume the change in length is small, and of the order $\frac{1}{\chi}$. Let us write,

$$
k\ell = k\ell_0 + \chi\delta \tag{5.18}
$$

hence

$$
\delta = \frac{k(\ell - \ell_0)}{\chi} = \frac{k\Delta\ell}{\chi}
$$

and

$$
\ell = \ell_0 + \frac{\chi\delta}{k} \tag{5.19}
$$

As we assumed a small change let us expand the relevant quantities in a power series of $\Delta\ell$. Equation (2.34) will become, after expanding the operator $K[v(\zeta), \zeta, \ell]$

$$V'' + k^2 V = \frac{g\pi\omega\mu x}{2}\left[H_t^{\circ} + K(V,\xi,l_\circ) + \Delta l \frac{\partial}{\partial l_\circ} K(V,\xi,l_\circ) + \cdots\right] \tag{5.20}$$

Expanding the boundary condition $V(\pm l) = 0$ into a power series gives us

$$V(\pm l_\circ) + \Delta l V'(\pm l_\circ) + \frac{1}{2}(\Delta l)^2 V''(\pm l_\circ) = 0 \tag{5.21}$$

As in Chapter II, we look for a solution in terms of powers of x.

$$V = V_0 + x V_1 + x^2 V_2 + \cdots \tag{5.22}$$

Substituting (5.22) into (5.20) we get a set of integro-differential equations. The solution for V_0 is as before, (2.37)

$$V_\circ = C_\circ \varphi(\xi) = C_\circ \begin{cases} \sin k\xi \\ \cos k\xi \end{cases} \tag{5.23}$$

To determine the amplitude, as in Chapter II, we multiply the equation for V_1 by V_0 and integrate from $-l$ to $+l$. Instead of its being equal to zero as in Chapter II, we find now

$$\int_{-l}^{l} (V_0 V_1'' + k^2 V_0 V_1)\, d\xi = 2k\sigma C_0^2 \tag{5.24}$$

Integrating the right side will give us an expression for C_0. We get

$$C_0 = \frac{\int_{-l}^{l} H_t^{\circ} \varphi(\xi)\, d\xi}{-\int_{-l}^{l} \varphi(\xi)\, K[\varphi(\xi), \xi, l_0]\, d\xi + j\, \frac{4K\delta}{\pi\mu\omega}} \tag{5.25}$$

We recognize immediately that the denominator in (5.25) is the new admittance function. The integral expression is identical with the function $-Y_n$ we have considered in Chapter IV. We see therefore, that the change is only in the imaginary part of the admittance. The real part is invariant to slight changes in length. If we rewrite the reactive part, from (4.47) and (5.25), we get after substitution of $\delta = \frac{k\Delta l}{x} = k\Delta l \ln kd$

$$B = \frac{1}{c\mu\pi}\left[S_i(2\pi) - \frac{2\pi}{ab} \sum_{\nu=1}^{N} \frac{k \cos^2\left(\frac{\nu\pi x}{a}\right) \sin\left(\beta_\nu \frac{\pi}{k}\right)}{\beta_\nu (k^2 - \beta_\nu^2)} + 4k\Delta l \ln kd \right] \tag{5.26}$$

Let us see what the order of reactance change is from the correction term $4k\, \Delta l \ln kd$. For a slot with a width of 0.030" we have

$$\ln 0.018 = -4.02$$

and

$$4k\Delta l \ln kd = -31.55\, \Delta l$$

For a change in length of 0.001" we have $\Delta l = 2.54 \; 10^{-3}$ cm. and the change in reactance is -0.08. Adding this to values in (5.9) we now get

$$-Y_n = \frac{1}{120\pi^2}\,(1.68 - j0.35)$$

and the phase angle will now be -168°. This is closer to the measured values.

It should be remembered that the accuracy of the measurement of phase is poor. In addition, from above it is seen that the phase is sensitive to slot length, and hence to frequency changes. For example, a change of 5 Mc.p.s. in 10,000 represents a change of the order of 0.001" in terms of slot length.

Further, the theory neglects the effects at the edges. We have neglected $E_{\zeta}(\eta,\zeta)$ in comparison with $E_{\eta}(\eta,\zeta)$, but at the edges of the slot $E_{\eta}(\eta,\zeta)$ goes to zero, whereas $E_{\zeta}(\eta,\zeta)$ is the component that does not have to disappear there. As we have seen that the phase is rather sensitive to the length of the slot, we should expect this edge effect to influence the phase results. Another factor which has been neglected is the finite thickness of the guide wall. To check the effect of slight changes in slot length, the phase as a function of frequency has been measured. The frequency varies over a small range around the "correct" frequency of 9375 MC. The results are plotted in Fig. 9. It can be seen from the experimental curve that for slight changes in frequency, there are rather appreciable changes in phase. With all these factors in mind the agreement in phase between the theoretical and experimental results is rather good.

Discussion of the Experiments

The experimental set up has been described before[16]. A schematic diagram is shown in Fig. 10. The principal unit is a mode transducer, designed so as to excite separately and independently either mode TE_{10} or mode TE_{20} in the double mode guide. In a former paper[17] we have shown the validity of measuring the scattering coefficient through such a transducer. This has been shown to be true only if the transducer is matched in every direction. Any deviation from the match will result in experimental errors. This sensitivity to a match in four ways results in a system which is rather frequency sensitive. Although care was taken in the matching work, the accuracy of the system is limited. The measuring equipment employed would yield results within $\frac{1}{2}$ db in amplitude and 5 degrees in phase. We have found, however, that the accuracy in amplitude is only within 1.5 db and the phase errors are about 15 degrees. The reliability of the phase measurements, in particular, is below what might be considered desirable. This is due to the sensitivity of the system to the joints in the microwave plumbing, and the fact that the measuring procedure required moving these joints. The elimination of cables, and the use of waveguides with rotating joints has improved that part considerably. The amplitudes and phases were measured in two ways. In the reflected wave region the method employed was the Voltage Standing Wave Ratio, and in the transmitted region, a magic T bridge with a compensating attenuator and phase shifter for balancing were employed. It should be pointed out that the techniques for measuring in multimode guides are just at the beginning of their development, and need to be developed.

Conclusions

From this investigation we can conclude that, for the particular case of a slot parallel to the axis of the guide and located on the broad face of the guide in a rectangular guide propagating the TE_{10} and TE_{20} modes a slot interacts mainly with the higher mode. This will be the case for any number of modes. If a slot is to be used as a radiator, the mode to launch in the guide is the highest one possible. This will give the maximum radiation power. From expression (5.1) it can be seen that the reason for this behavior is the fact that the guide wavelength for lower modes is close to the free space wavelength. The factor $\cos^2\beta \ell$ is very close to zero. Realizing that, it is evident that slots could be used as a means of separating the modes.

Although the expressions derived here are for a resonant slot, they indicate the behavior of slots in general. We have seen that for small changes in length the resistive part is invariant. It can be safely assumed that the general character of slots of length not too different from the resonant length would not be appreciably different from what we found for the resonant length. One can then find such a length of longitudinal slot that will interact with one mode only. As such, these slots would serve as mode separators. One could also have two slots arranged so that one interacts with one mode and the other with the second mode. This could be applied of course, to any number of modes and slots. Such an arrangement will serve as a launcher, or a receiving filter, for a multiplex system. Every mode can be made to carry another message in the same waveguide.

From the theory developed here we can arrive at some idea for investigating scattered fields in general. The major part of the information is derivable from the singularity of the source functions. From the theory of functions this is to be expected. Nevertheless, this way of attack has been neglected in comparison with the modal expansion method. The modal expansion method can claim more rigour in writing out a formal solution. If no recourse is taken to approximation procedures, it gives a solution in terms of infinite series. In most cases the convergence of these series leaves much to be desired. It would probably be fruitful to investigate the region of singularity first. For electromagnetic fields the singularity of the source functions is well known, and its location is in the region of integration. Separating it out, and treating it separately will give the major part of the wanted information. The special nature of the region of interest can then be introduced as a correction, or a refinement of the information derived from the singularity.

Suggestions for Further Research

Several problems amenable to treatment by the outlined method present themselves. In principle there is no difficulty in extending the method presented here to slots of arbitrary length. It will involve a more complicated integration, but this could be evaluated either approximately or rigorously. One would also want to know in detail the behavior of slots at an arbitrary angle to the direction of propagation. Another important problem in the design of slot arrays is the interaction of adjacent slots. With the expressions for the voltage induced in a slot by an arbitrary exciting field this problem could be solved without much difficulty. A class of problems of a little different nature are diffraction problems in free space. It would be of interest to apply the outlined method to the problems of diffraction.

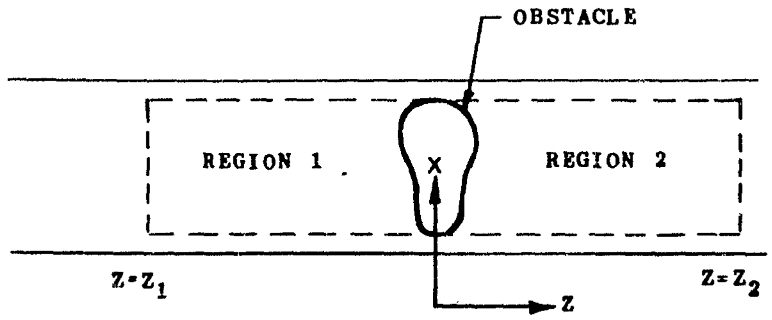

FIG. 1

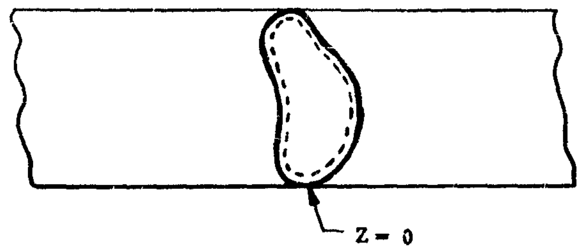

FIG. 2

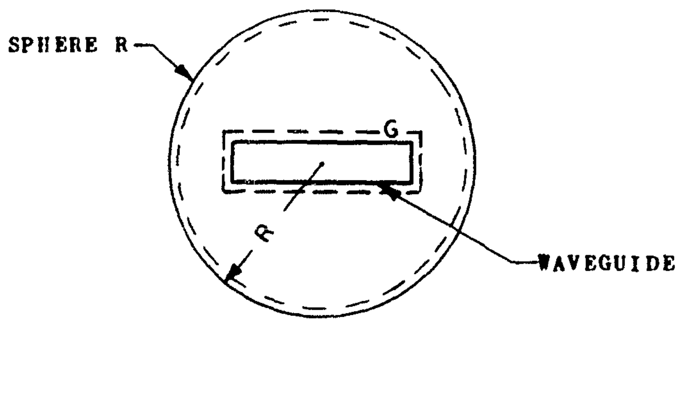

FIG. 3

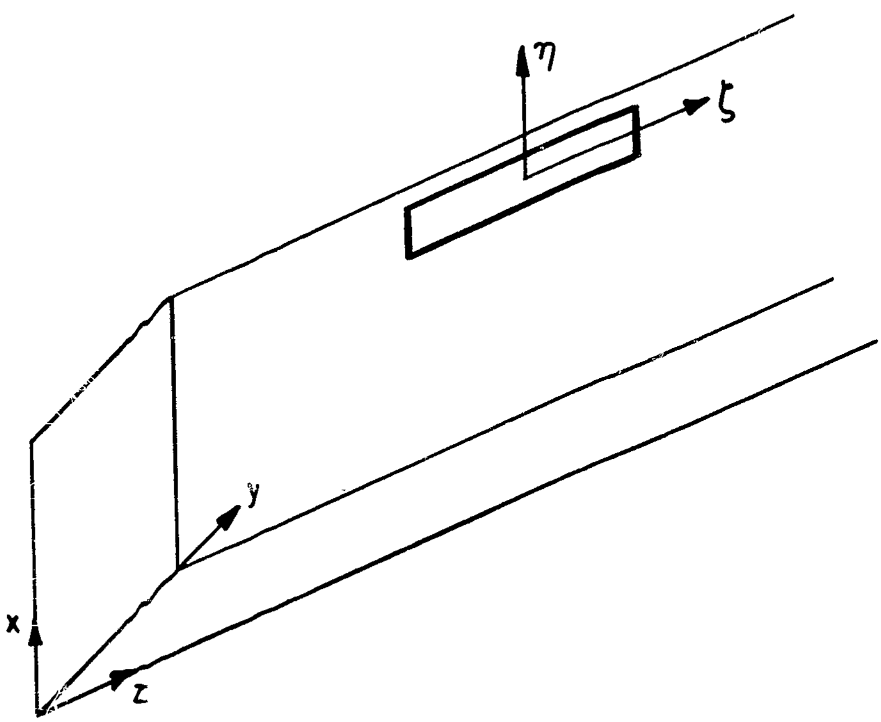

FIG. 4

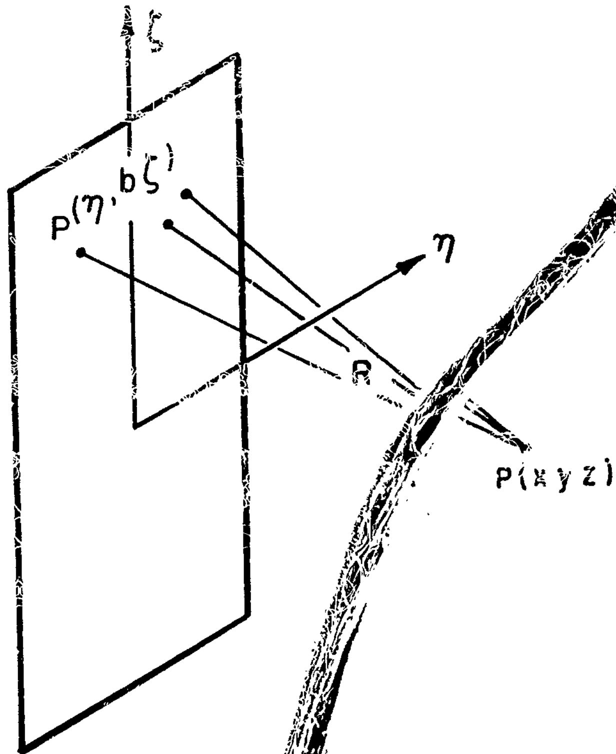

FIG. 5

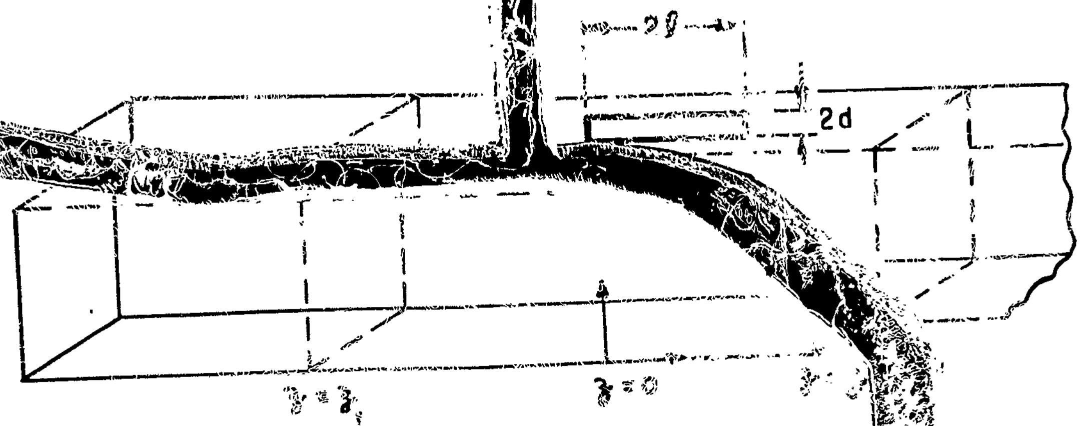

FIG. 6

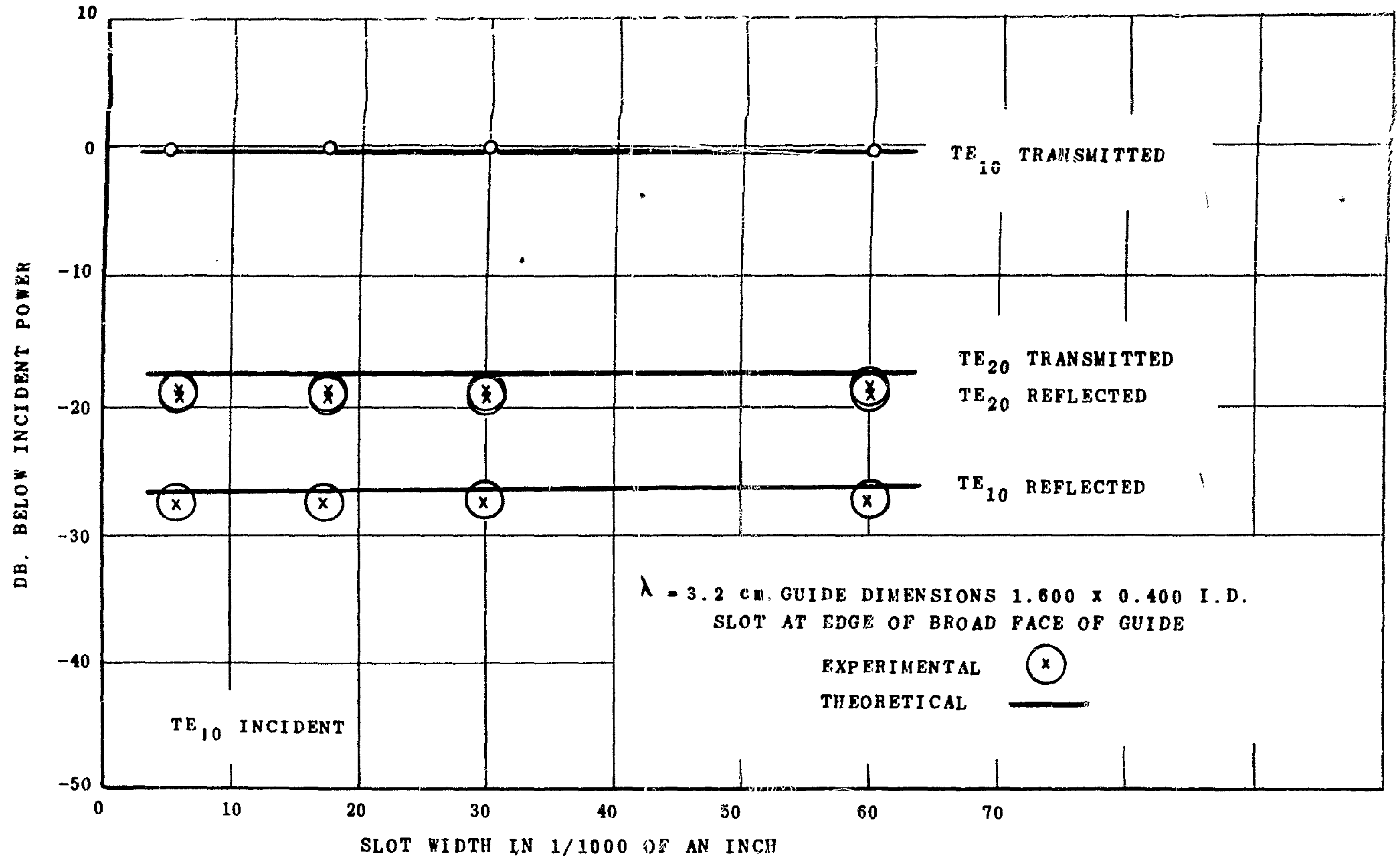

FIG. 7 TRANSMISSION AND REFLECTION COEFFICIENTS FOR A HALF WAVELENGTH LONGITUDINAL SLOT

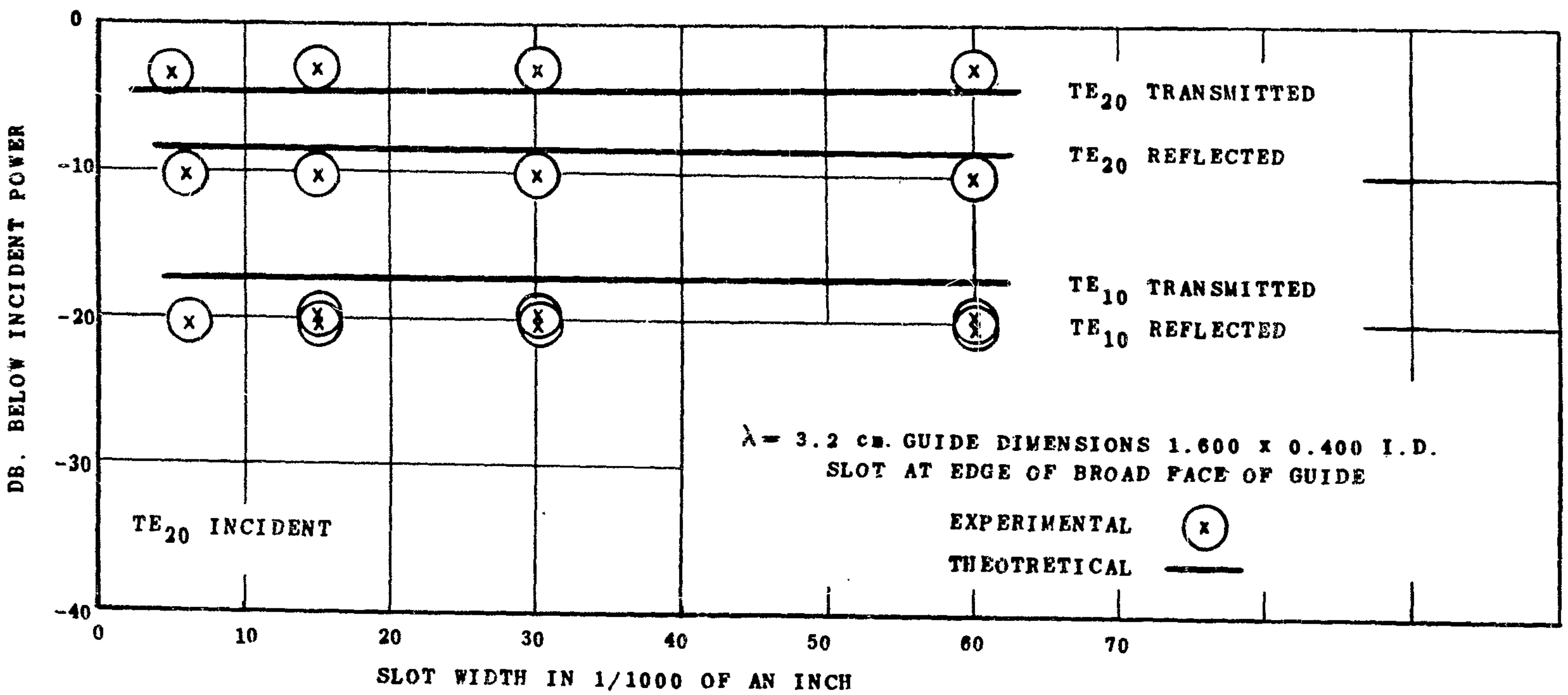

FIG. 8 TRANSMISSION AND REFLECTION COEFFICIENTS FOR A HALF WAVELENGTH LONGITUDINAL SLOT

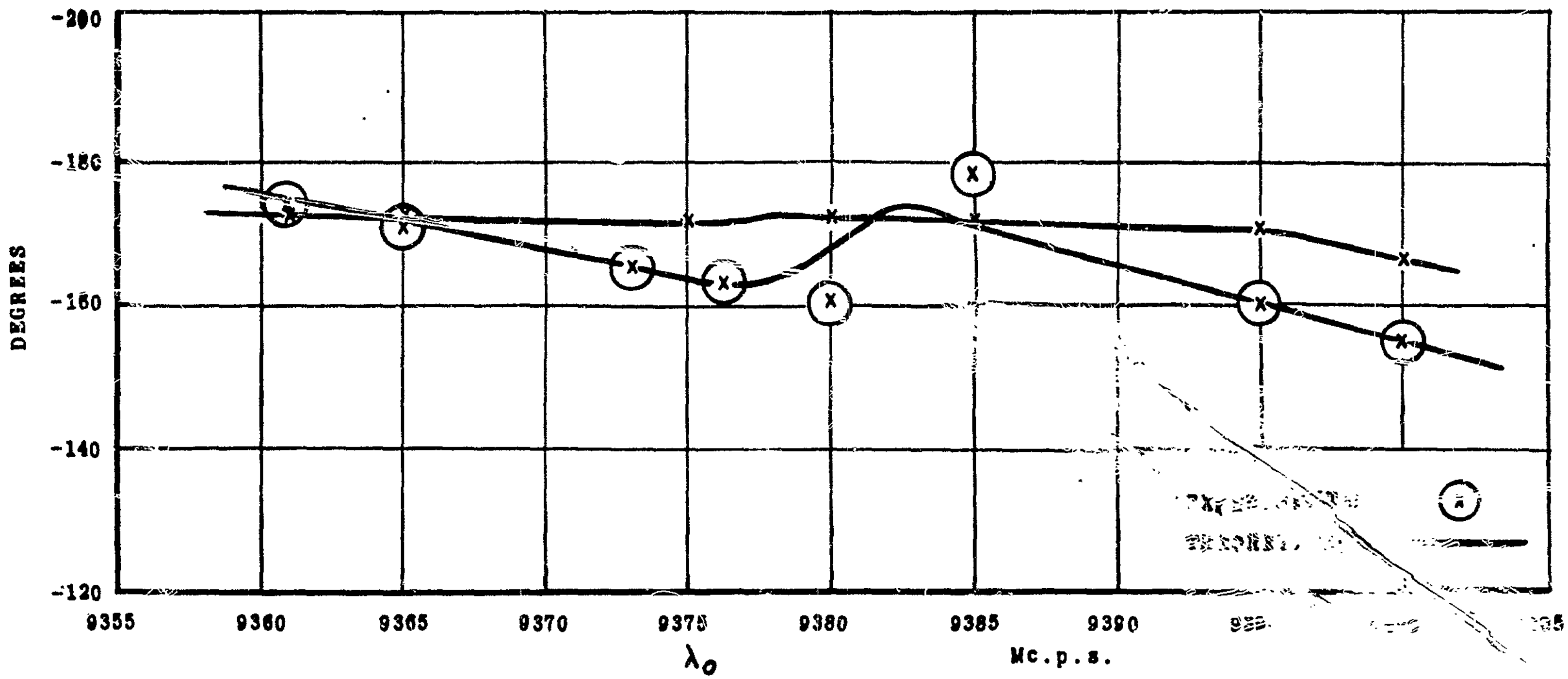

FIG. 9 PHASE VARIATION OF REFLECTION COEFFICIENT AROUND RESONANT FREQUENCY.

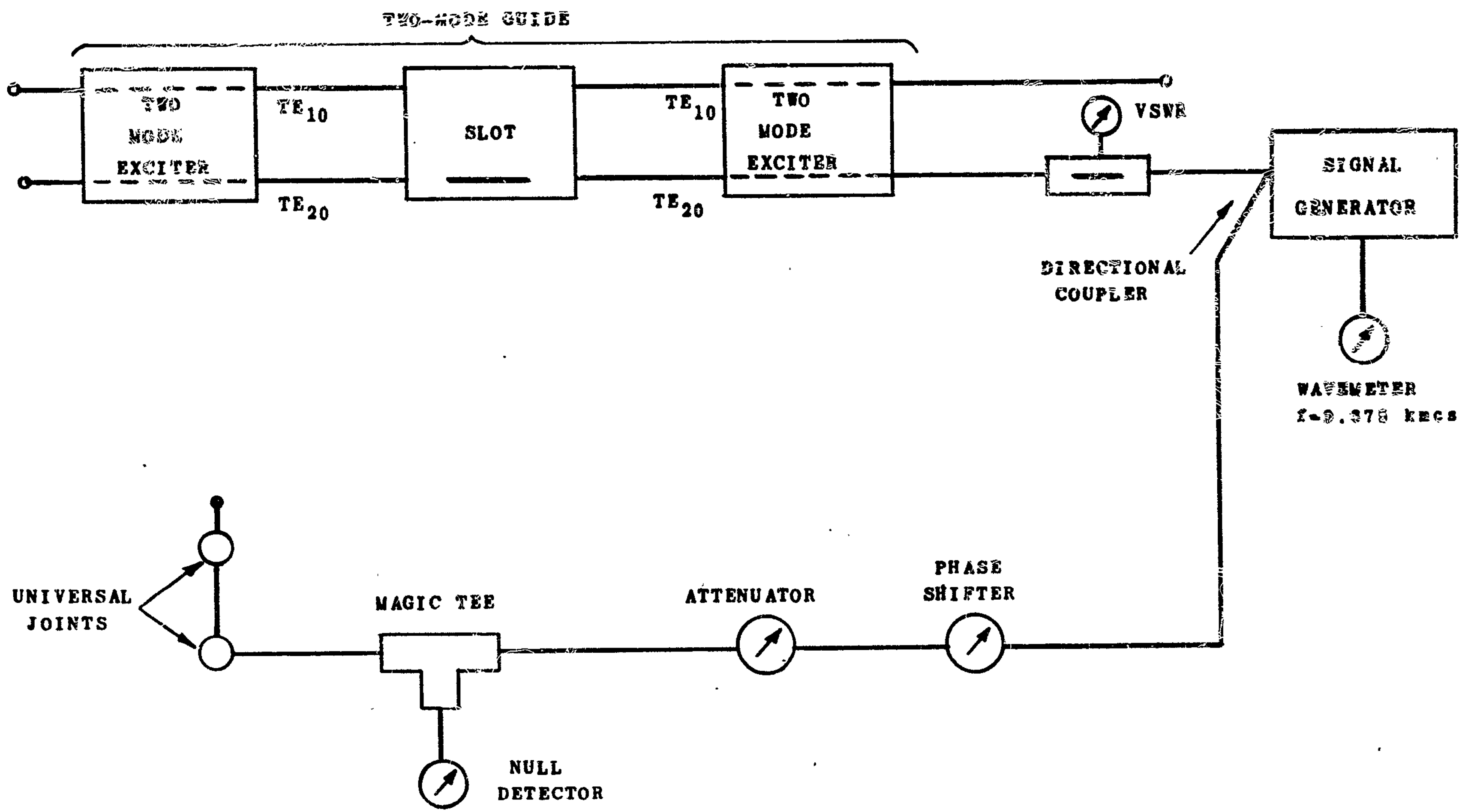

FIG. 10 SCHEMATIC LAYOUT FOR MEASURING TRANSMISSION AND REFLECTION COEFFICIENTS WITH TE_{20} INCIDENT ON SLOT

APPENDIX 1

In the transformation leading to equation (1.18) the following is required.

$$\frac{\omega}{2}\iint_{\sigma}\left(\underline{e}_k^- \times \underline{h}_k^+ - \underline{e}_k^+ \times \underline{h}_k^-\right)\cdot \underline{n}\, d\sigma \qquad \text{(A1.1)}$$
$$z = z_1$$

Referring to (1.17) and (1.18) we get

$$T_{z_1} = \iint\left(\underline{\psi}_k^- \times \underline{\varphi}_k^+ - \underline{\psi}_k^+ \times \underline{\varphi}_k^-\right)\cdot n\, d\sigma \qquad \text{(A1.2)}$$

Consider the first term under the integral. For the cross product of the vector mode functions we have from (1.8)

$$\iint \underline{\psi}_k^- \times \underline{\varphi}_k^+ \cdot \underline{n}\, dx\,dy = \iint\left(\psi_{kx}^- \varphi_{ky}^+ - \psi_{ky}^- \varphi_{kx}^+\right) dx\,dy =$$

$$\iint\left\{\left[-j\sqrt{\frac{2\mu}{\beta_k\omega}}\frac{\partial u}{\partial y}\, j\sqrt{\frac{2\beta_k}{\omega\mu}}\frac{\partial u}{\partial y}\right] - \left[j\sqrt{\frac{2\mu}{\beta_k\omega}}\frac{\partial u}{\partial x}\, j\sqrt{\frac{2\beta_k}{\omega\mu}}\frac{\partial u}{\partial x}\right]\right\} dx\,dy =$$

$$\iint\frac{2}{\omega}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy = \frac{2}{\omega} \qquad \text{(A1.3)}$$

With similar substitutions we find for the second term in (A1.2)

$$\iint \underline{\psi}_k^+ \times \underline{\varphi}_k^- \cdot \underline{n}\, dx\,dy = -\frac{2}{\omega} \qquad \text{(A1.4)}$$

Adding these two we find that

$$T_{z_1} = \frac{\omega}{2}\cdot\frac{4}{\omega} = 2 \qquad \text{(A1.5)}$$

In a similar fashion one can find that when direction of normal vector is reversed, we get for the integral over the cross-section the value of (-2). With the help of such relations the indicated evaluation in Chapter I can be carried out.

APPENDIX 2

To compute the transformation from (4.13) to (4.14) consider the expression

$$u'' + k^2 u = \left(\frac{\partial^2}{\partial z^2} + k^2\right)\Big[\int_{-l}^{z} \ln 2k(z-z')\left[\varphi'(z') + jk\varphi(z')\right]e^{-jk(z-z')}dz' -$$

$$\int_{z}^{l} \ln 2k(z'-z)\left[\varphi'(z') - jk\varphi(z')\right]e^{-jk(z'-z)}dz'\Big] \qquad \text{(A2.1)}$$

Each of the two integrals in (A2.1) is regular, and we can differentiate them as functions of z. We have

$$\frac{\partial}{\partial z}\int_{a}^{z} F(z,z')\,dz' = F(z,z) + \int_{a}^{z} \frac{\partial F(z,z')}{\partial z}\,dz' \qquad \text{(A2.2)}$$

and we have to evaluate the integrals in (A2-2). For this purpose we introduce transformations as follows:

$$\ln 2k(z-z') = u_1(z,z')$$
$$\ln 2k(z'-z) = u_2(z,z') \qquad \text{(A2.3)}$$

and note that

$$\frac{\partial u_1}{\partial z} = -\frac{\partial u_1}{\partial z'} \quad ; \quad \frac{\partial^2 u_1}{\partial z^2} = \frac{\partial^2 u_1}{\partial z'^2}$$
$$\frac{\partial u_2}{\partial z} = -\frac{\partial u_2}{\partial z'} \quad ; \quad \frac{\partial^2 u_2}{\partial z^2} = \frac{\partial^2 u_2}{\partial z'^2} \qquad \text{(A2.4)}$$

Further denote

$$e^{-jk(z-z')} = f_1(z,z') \qquad \text{(A2.5)}$$

$$e^{-jk(z'-z)} = f_2(z,z')$$

and note that

$$\frac{\partial f_1}{\partial z} = -\frac{\partial f_1}{\partial z'} \quad ; \quad \frac{\partial^2 f_1}{\partial z^2} = \frac{\partial^2 f_1}{\partial z'^2} \qquad \text{(A2.6)}$$

$$\frac{\partial f_2}{\partial z} = -\frac{\partial f_2}{\partial z'} \quad ; \quad \frac{\partial^2 f_2}{\partial z^2} = \frac{\partial^2 f_2}{\partial z'^2}$$

Also substitute

$$\varphi'(z') \pm jk\varphi(z') = P_{1,2}(z') \qquad \text{(A2.7)}$$

We can write now

$$u'' + k^2 u = L_1(z) - L_2(z) \qquad \text{(A2.8)}$$

where

$$L_1(z) = \int_{l}^{z} \left(\frac{\partial^2}{\partial z^2} + k^2\right)\left[u_1(z,z') f_1(z,z') P_1(z')\right] dz' + N_1(z) \qquad \text{(A2.9)}$$

and

$$L_2(z) = \int_{z'}^{l} \left(\frac{\partial^2}{\partial z^2} + k^2\right)\left[u_2(z,z') f_2(z,z') P_2(z')\right] dz' + N_2(z) \qquad \text{(A2.10)}$$

Consider $L_1(z)$

$$L_1(z) - N_1 = \int_{-l}^{z} \left(\frac{\partial^2 u}{\partial z^2} \gamma_1 \rho_1 + 2 \frac{\partial u_1}{\partial z} \frac{\partial \gamma_1}{\partial z} \rho_1 + u_1 \frac{\partial^2 \gamma_1}{\partial z^2} \rho_1 + k^2 u_1 \gamma_1 \rho_1 \right) dz' \tag{A2.11}$$

Bu (A2.4) and (A2.6) we can rewrite this in the form

$$\int_{-l}^{z} \left(\frac{\partial^2 u}{\partial z'^2} \gamma_1 \rho_1 + 2 \frac{\partial u_1}{\partial z'} \frac{\partial \gamma_1}{\partial z'} \rho_1 + u_1 \frac{\partial^2 \gamma_1}{\partial z'^2} \rho_1 + k^2 u_1 \gamma_1 \rho_1 \right) dz' =$$

$$\int_{-l}^{z} \left[\frac{\partial}{\partial z'} \left(\frac{\partial u}{\partial z'} \gamma_1 \rho_1 \right) - \frac{\partial}{\partial z'} (u_1 \gamma_1) \frac{\partial \rho_1}{\partial z'} + \frac{\partial}{\partial z'} \left(u_1 \frac{\partial \gamma_1}{\partial z'} \rho_1 \right) + k^2 u_1 \gamma_1 \rho_1 \right] dz' \tag{A2.11a}$$

The first and third term can be taken out of the integral directly. The second term we integrate by parts. Thus we get

$$L_1(z) - N_1 = \frac{\partial u_1}{\partial z'} \gamma_1 \rho_1 \Big|_{-l}^{z} + u_1 \frac{\partial \gamma_1}{\partial z'} \rho_1 \Big|_{-l}^{z} - u_1 \gamma_1 \frac{\partial \rho_1}{\partial z'} \Big|_{-l}^{z} + \int_{-l}^{z} \left(u_1 \gamma_1 \frac{\partial^2 \rho}{\partial z^2} + k^2 u_1 \gamma_1 \rho_1 \right) dz' \tag{A2.12}$$

In a similar fashion we can treat $L_2(z)$. We then get

$$L_2(z) - N_2 = \frac{\partial u_2}{\partial z'} \gamma_2 \rho_2 \Big|_{z}^{l} + u_2 \frac{\partial \gamma_2}{\partial z'} \Big|_{z}^{l} - u_2 \gamma_2 \frac{\partial \rho_2}{\partial z'} \Big|_{z}^{l} + \int_{z}^{l} \left(u_2 \gamma_2 \frac{\partial^2 \rho_2}{\partial z'^2} + k^2 u_2 \gamma_2 \rho_2 \right) dz' \tag{A2.13}$$

Substituting back from (A2.3), (A2.5) and (A2.7) and $\varphi(\pm l) = 0$ we find that

$$L_1 - L_2 = \frac{\varphi'(-\ell)}{\ell + z} e^{-jk(\ell+z)} - \frac{\varphi'(+\ell)}{\ell - z} e^{-jk(\ell-z)} + \varphi''(-\ell) \ln 2k(\ell+z) e^{-jk(\ell+z)}$$
$$+ \varphi''(+\ell) \ln 2k(\ell-z) e^{-jk(\ell-z)} + \int_{-\ell}^{z} \ln 2k(z-z') e^{-jk(z-z')} [\varphi''' + jk\varphi'' + k^2\varphi' + jk^3\varphi] dz'$$
$$- \int_{z}^{\ell} \ln 2k(z'-z) e^{-jk(z'-z)} [\varphi''' - jk\varphi'' + k^2\varphi' - jk^3\varphi] dz' \qquad \text{(A2.14)}$$

In our case we have

$$\varphi(z') = \cos kz \qquad \text{(A2.15)}$$

hence

$$\varphi'(-\ell) = k \quad ; \quad \varphi'(+\ell) = -k \quad \text{as} \quad k\ell = \frac{\pi}{2} \qquad \text{(A2.16)}$$

and

$$\varphi''(\pm \ell) = 0 \qquad \text{(A2.17)}$$

Further in our case

$$\varphi'''(z') + k^2\varphi'(z') = \frac{\partial}{\partial z'}(\varphi'' + k^2\varphi) = 0 \qquad \text{(A2.18)}$$

and

$$\pm jk[\varphi''(z') + k^2\varphi(z')] = 0 \qquad \text{(A2.19)}$$

Substituting (A2.10) to (A2.19) in (A2.14) and adding the integrated terms from (A2.2) we find

$$u'' + k^2 u = k\left[\frac{e^{-jk(l+z)}}{l+z} + \frac{e^{-jk(l-z)}}{l-z}\right] \tag{A2.20}$$

which is expression (4.14) used in text.

APPENDIX 3

The transformation from (4.35) to (4.36) involves computation of the expression

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)S\int_{-l}^{l} e^{j\beta|z-z'|}\,\varphi(z')\,dz' = P_i'' + k^2 P_i \tag{A3.1}$$

where

$$\varphi(z') = \cos kz' \tag{A3.2}$$

To compute this we separate the integral into two parts and write

$$m = \left(\frac{\partial^2}{\partial z^2}+k^2\right)\int_{-l}^{z} e^{j\beta(z-z')}\cos kz'\,dz' + \left(\frac{\partial^2}{\partial z^2}+k^2\right)\int_{z}^{l} e^{j\beta(z'-z)}\cos kz'\,dz' \tag{A3.3}$$

Each of these two integrals can be evaluated directly. We have

$$\int_{-l}^{z} e^{j\beta(z-z')}\cos kz'\,dz' = e^{j\beta z}\left.\frac{e^{-j\beta z'}(-j\beta\cos kz' + k\sin kz')}{k^2-\beta^2}\right|_{-l}^{z}$$

$$= \left[e^{-j\beta z}\left(\frac{-j\beta\cos kz + k\sin kz}{k^2-\beta^2}\right) - e^{j\beta l}\left(\frac{-k}{k^2-\beta^2}\right)\right]e^{j\beta z} \tag{A3.4}$$

Further

$$\int_{z}^{l} e^{j(z-z')}\cos kz'\,d = \left[e^{j\beta l}\frac{k}{k^2-\beta^2} - e^{j\beta z}\left(\frac{j\beta\cos kz + k\sin kz}{k^2-\beta^2}\right)\right]e^{-j\beta z} \tag{A3.5}$$

Adding (A3.4) and (A3.5) and substituting in (A3.3) we find

$$M = \left(\frac{\partial^2}{\partial z^2}+k^2\right)\left[\frac{k}{k^2-\beta^2}e^{j\beta(l+z)}+e^{j\beta(l-z)}\right]-\frac{2j\beta}{k^2-\beta^2}\cos kz \tag{A3.6}$$

This can be computed directly. We get

$$M = \Big[\left(e^{j\beta(l+z)}+e^{j\beta(l-z)}\right)\left(\frac{k(-\beta^2)}{k^2-\beta^2}\right)+\frac{2j\beta k^2}{k^2-\beta^2}\cos kz +$$
$$\left(e^{j\beta(l+z)}+e^{j\beta(l-z)}\right)\frac{k\,k^2}{k^2-\beta^2}-\frac{2j\beta k^2}{k^2-\beta^2}\cos kz\Big] \tag{A3.7}$$

Simply adding up the terms will give us

$$M = k\left(e^{j\beta(l+z)}+e^{j\beta(l-z)}\right) \tag{A3.8}$$

Which is the expression given in the text.

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